# Lecture notes

July 10, 2023

# Contents

# I (Review) Lecture 1: Some category theory

Several people said they wanted to learn basic category theory. We already needed to review some category theory for this seminar, so we'll take the opportunity to review enough theory to hopefully help everyone.

The short-term plan is to review some category theory, Grothendieck sites, and sheaves. Then we will start actually talking about condensed things. I only reluctantly call this "review," since we are *not* assuming everyone is familiar with all the material moving forward. The goal is only to catch folks up to a working knowledge.

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Probably everything we say here is said better somewhere in Emily Riehl's *Category Theory in Context*. That book also has many examples. You should read it.

**Definition I.1.** A *category* $\mathsf{C}$ consists of the following data.

(1) A collection $Ob(\mathsf{C})$ we call *objects*.

(2) A collection $Mor(\mathsf{C})$ we call *morphisms*.

(3) For each morphism $f$, a *source* and *target* object. (We write $f : X \to Y$ to express that $f$ is a morphism with source $X$ and target $Y$.)

(4) For each object $X$, a distinguished morphism $\mathrm{id}_X : X \to X$ we call the *identity morphism*.

(5) For each pair of morphisms $f : X \to Y, g : Y \to Z$ such that $target(f) = source(g)$, a distinguished morphism $gf : X \to Z$ we call the *composite morphism*.

And this data must satisfy the following properties.

- Given a morphism $f : X \to Y$, we have $f = \mathrm{id}_Y f = f\mathrm{id}_G$.
- Given morphisms $f, g, h$, we have $(fg)h = f(gh)$ (when the source/targets match appropriately).

In practice, we think of categories as "like a collection of objects and maps between them, with all the structure that should accompany the word *maps*—identity self-maps, composites, associativity."

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## I.1 Examples of categories

(See Riehl for more)

**Example I.2.** The category of sets $\mathsf{Set}$ has sets as objects and functions as morphisms. The rest of the structure is "obvious": source/target is domain/range, the composition of morphisms is defined as the composition of functions, and identity morphisms are the identity functions.

**Example I.3.** The category of spaces $\mathsf{Top}$ has topological spaces as objects and continuous maps as morphisms. Again, the rest of the structure is "obvious."

**Example I.4.** Define the category $\mathsf{Top}_*$ to have *based spaces*[1] as objects and *based maps*[2] as morphisms.

**Example I.5.** Define the category $Grp$ to have groups as objects and homomorphisms as morphisms.

**Example I.6.** Define the category $\mathsf{Ab}$ to have abelian groups as objects and homomorphisms as morphisms.

**Example I.7.** Denote by $k$ a field (e.g., $k = \mathbb{R}$). Define the category $Vect_k$ to have $k$-vector fields as objects and $k$-linear maps as morphisms.

**Example I.8** (Morphisms do not have to be functions!)**.** Define a category $\mathsf{Naturals}$ to have

- As objects, the natural numbers $Ob(\mathsf{Naturals}) := \{0, 1, 2, \dots\}$; and

[1] A based space is pair $(X, x)$ where $X$ is a space and $x \in X$.
[2] A based map $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$.

- A morphism $a \to b$ for each pair of numbers $(a, b)$ such that $a \leq b$.

Thus, given objects $a, b \in \{0, 1, 2, \dots\}$, there is at most one arrow $a \to b$, and it exists iff $a \leq b$.

**Example I.9** (Morphisms do not have to be functions!). Define a category $\mathsf{Skel}(\mathsf{FinSet}_{\mathsf{iso}})$ to have

- Natural numbers $n$ as objects; and
- A morphism $n \to n$ for each element of the symmetric group $\Sigma_n$, and no morphisms $m \to n$ when $m \neq n$.

**Example I.10** (Example from Peter's talk). A *poset* is a set $S$ with a relation $\leq$ that is reflexive, transitive, and antisymmetric. A *morphism of posets* $f : (S, \leq) \to (S', \leq')$ is a function $f : S \to S'$ that respects the partial orderings, i.e. $x \leq y \iff f(x) \leq' f(y)$. We denote by $Poset$ the category of posets and morphisms of posets.

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## I.2 Isomorphisms

All sorts of objects—groups, rings, sets, spaces—have a notion of "sameness." In an arbitrary category, we formalize this notion as *isomorphisms*.

**Definition I.11.** Let $\mathsf{C}$ be a category. Suppose that $f : c \to c'$ is a morphism and that there exists a morphism $g : c' \to c$ such that $fg = \mathrm{id}_{c'}$ and $gf = \mathrm{id}_c$. Then we say $f$ and $g$ are *isomorphisms* and we say that $c, c'$ are *isomorphic*.

**Example I.12.** Isomorphisms in $\mathsf{Set}$ are bijections. Isomorphisms in $\mathsf{Top}$ are homeomorphisms. Isomorphisms in $\mathsf{Top}_*$ are based homeomorphisms. Isomorphisms in $Grp$ and $\mathsf{Ab}$ are group isomorphisms.

Exercise: Figure out what the isomorphisms in $Poset$ are.

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## I.3 Functors

**Definition I.13.** Let $\mathsf{C}, \mathsf{D}$ be categories. A *functor from $\mathsf{C}$ to $\mathsf{D}$* (which we write $F : \mathsf{C} \to \mathsf{D}$) is "a map of objects and morphisms that preserves categorical structure, i.e. sources, targets, composites, and identities." Formally, it consists of the following data.

(1) An object $FX \in \mathsf{D}$ for each object $X \in \mathsf{C}$.

(2) For each morphism $f : X \to Y$ in $\mathsf{C}$, a morphism $Ff : FX \to FY$ in $\mathsf{D}$.

And this data must satisfy the following properties.

- For any pair of composable morphisms $f, g$ in $\mathsf{C}$, we have $F(gf) = F(g)F(f)$.
- For any identity morphism $\mathrm{id}_X$ in $\mathsf{C}$, we have $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.

Very often, we say "(one type of object) are the same thing as (another type of objects)." Categories give us a great, concrete way to talk about "types of objects." Functors give us a way to "modify and compare" objects of different types. Can functors tell us when (one type of object) are "the same thing as" (another type)? Yes, and this is a very useful notion.

For the following definition, we will denote by $Mor(X, Y)$ the set of morphisms $X \to Y$ between objects $X, Y \in \mathsf{C}$.

**Definition I.14.** Let $\mathsf{C}, \mathsf{D}$ be categories. An *equivalence of categories* is a functor $F : \mathsf{C} \to \mathsf{D}$ that is

1. *Full*: for every pair of objects $X, Y \in \mathsf{C}$, the mapping $f \mapsto F(f)$ defines a surjection $Mor(X, Y) \to Mor(FX, FY)$;
2. *Faithful*: for every pair of objects $X, Y \in \mathsf{C}$, the mapping $f \mapsto F(f)$ defines an injection $Mor(X, Y) \to Mor(FX, FY)$; and

3. *Essentially surjective*: for every object $d \in \mathsf{D}$, there exists some $c \in \mathsf{C}$ such that $Fc \cong d$.

**Remark I.15.** The analogy is, "full and faithful is like injectivity" and "essentially surjective is like surjectivity." If you have both, you have an isomorphism. Note that a full and faithful functor need not actually be surjective *on objects*.

**Remark I.16.** There are other common, equivalent definitions of an equivalence of categories.

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## I.4 Examples of functors

(see also Riehl, p. 13)

**Example I.17.** Let $\mathsf{C}$ denote one of the categories $\mathsf{Top}, Grp, \mathsf{Ab}$, or $Vect_k$. We can define a functor $U : \mathsf{C} \to \mathsf{Set}$ by mapping objects to their underlying sets and morphisms to their underlying set-maps. We call $U$ the *forgetful functor*. (We generally refer to any functor that "tosses out" structure, e.g. a topology on a set, as a *forgetful functor*.)

**Example I.18.** If $\mathsf{C}$ is any category, then we can form its *opposite category* $\mathsf{C}^{\mathrm{op}}$ to have the same objects but with "flipped arrows," i.e. swapped source/targets of $\mathsf{C}$'s morphisms. There is a functor $\mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ that takes objects to themselves and morphisms to their "flip."

Exercise: prove that $(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ is equivalent to $\mathsf{C}$.

**Example I.19.** For $V \in Vect_k$, recall that its *dual* is defined as the vector space $V^* := \{\text{linear maps } V \to k\}$. Given a linear map $f : V \to W$, there is induced a map $f^* : W^* \to V^*$ that sends $v : W \to k$ to $v \circ f : V \to k$. The mapping $V \mapsto V^*, f \mapsto f^*$ defines the *dualization functor* $(-)^* : Vect_k \to Vect_k^{\mathrm{op}}$.

**Remark I.20.** You have heard probably heard that we have an isomorphism $V \cong V^{**}$ that is "canonical" or "natural" or "very nice," but that we do not have such an isomorphism $V \cong V^*$. (Although the two are isomorphic.) This can be expressed very concretely as a statement about the functor $(-)^*$ and its self-composite $(-)^{**}$. We do not yet have the language for this (natural transformations); the non-categorical reason is that an isomorphism $V \cong V^*$ requires a choice of basis, but there is an isomorphism $V \cong V^{**}$ that does not need any choice.

Many—and historically, the motivating—examples of functors come from algebraic topology.

**Example I.21.** Let $(X, x)$ be a based space (i.e., $X$ is a space and $x \in X$). We define the *fundamental group $\pi_1(X, x)$* as the set of based continuous maps $\ell : [0, 1] \to X$ such that $\ell(0) = \ell(1) = x$, modulo homotopy equivalence. The group structure is loop concatenation: given $\ell, \ell' : [0, 1] \to X$, define $\ell'\ell : [0, 1] \to X$ to do one loop over $[0.5]$ then the other over $[0.5, 1]$. Given a based map $f : (X, x) \to (Y, y)$, there is induced a map $f_* : \pi_1 X \to \pi_1 Y$ given by $\ell \mapsto f \circ \ell$. This defines a functor

$$\pi_1(-) : \mathsf{Top}_* \to \mathsf{Grp}.$$

**Example I.22.** For each $n$, singular homology defines a functor $H_n(-) : \mathsf{Top} \to \mathsf{Ab}$. Similarly, singular cohomology defines a functor $H^n(-) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Ab}$.

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## I.5 Natural transformations

We often want to compare *functors*. This will help us explain why e.g. "an arbitrary vector space is not *naturally* isomorphic to its dual" but "an arbitrary vector space *is* naturally isomorphic to its double-dual."

There are more serious examples where we *really* care about comparisons between functors. For example, the *Hurewicz homomorphism* from algebraic topology is a comparison $h_X : \pi_n(X) \to H_n(X)$ for every space $X$. But in fact, more can be said—for any continuous based map $f : X \to Y$, the Hurewicz homomorphism satisfies $h_Y \circ \pi_n(f) = H_n(f) \circ h_X$. (Here, $\pi_n(f)$ and $H_n(f)$ are the maps induced by $f$ on $pi_n$ and $H_n$.) This is a seriously useful fact that is not "formally guaranteed" to be true. One might phrase this as, "the Hurewicz homomorphism compares objects $\pi_n(X) \to H_n(X)$ in a way that respects how maps induce homomorphisms via the functors $\pi_n(-), H_n(-)$."

*Natural transformations* give a simple way to express this.

**Definition I.23.** Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. A *natural transformation from $F$ to $G$*, which we denote as $\alpha : F \implies G$, is the data of

- For each object $c \in \mathsf{C}$, a morphism $\alpha_c : F(c) \to G(c)$ in $\mathsf{D}$

such that for every morphism $f : c \to c'$ in $\mathsf{C}$, one has $G(f) \circ \alpha_c = \alpha_{c'} \circ F(f)$. In other words, the following diagram commutes.

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\alpha_c"] & F(c') \arrow[d, "\alpha'_c"] \\ G(c) \arrow[r, "G(f)"] & G(c') \end{tikzcd}$$

It turns out, natural transformations can be composed!

**Definition I.24.** Let $F, G, H : \mathsf{C} \to \mathsf{D}$ be three functors, and suppose we are given two natural transformations $\eta : F \Rightarrow G$ and $\varepsilon : G \Rightarrow H$. Define the *composition* $\varepsilon \circ \eta$ whose component at some $c \in \mathsf{C}$ is given by $(\varepsilon \circ \eta)_c := \varepsilon_c \circ \eta_c$.

**Exercise I.25.** Show that given natural transformations $\eta : F \Rightarrow G$ and $\varepsilon : G \to H$, that the composition $\varepsilon \circ \eta$ as defined above is actually a natural transformation. In other words, verify that the naturality condition is satisfied.

**Exercise I.26.** Show that given two categories $\mathsf{C}$ and $\mathsf{D}$, you can define the **functor category** $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ (also sometimes denoted as $[\mathsf{C}, \mathsf{D}]$ or $\mathsf{D}^{\mathsf{C}}$) whose objects are functors $\mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations $\eta : F \Rightarrow G$.

You will need to check that:

- Given any functor $F$ in $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$, there exists an **identity natural transformation** $\mathrm{Id}_F : F \Rightarrow F$.
- Composition of natural transformations is both associative and unital with respect to your identity transformations.

# II (Review) Lecture 2: More category theory

The first part of the talk on limits and colimits (sections 0.2.1 – 0.2.3 below) will be given by Isaiah (who is also writing the notes for these sections). If you have any questions about anything below, please do not hesitate to reach out to me on discord, my username there is isaiahtx.

## II.1 Reminder about natural transformations

First, I will review the definition of a natural transformation. If I have time, at some point during the presentation I will introduce the idea of a *category associated to a preorder*:

**Definition II.1.** A **preorder** is pair $(P, \leq)$ where $P$ is a set and $\leq$ is a reflexive and transitive relation on $P$, i.e., given $x \in P$, $x \leq x$, and if $x, y, z \in P$ satisfy $x \leq y$ and $y \leq z$, then $x \leq z$.

**Definition II.2.** A **preorder** is a category such that there is at most one morphism between any two objects.

**Exercise II.3.** Understand that the above two definitions are equivalent. In other words, show that given a preorder defined as in Definition II.1 gives rise to a preorder as defined in Definition II.2, and vice-versa.

## II.2 Limits and colimits

**Definition II.4.** Let $c$ be an object in a category $\mathsf{C}$.

We say $c$ is *initial* if, given any object $c'$ in $\mathsf{C}$, there is a unique morphism $c \to c'$.

Conversely, we say $c$ is *terminal* if, given an object $c'$ in $\mathsf{C}$, there is a unique morphism $c' \to c$.

**Definition II.5.** Recall: an arrow $f : x \to y$ in a category $\mathsf{C}$ is called an **isomorphism** if there exists an arrow $g : y \to x$ such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. We say two objects are **isomorphic** if there exists an isomorphism between them.

**Exercise II.6.** Show that in a category $\mathsf{C}$, given two initial objects $c$ and $c'$, there is a unique isomorphism $c \to c'$. Similarly, terminal objects in $\mathsf{C}$ are unique up to unique isomorphism. Thus, it makes sense to talk about *the* initial/terminal object in a category.

*Hint:* If $c$ and $c'$ are initial objects, there is a unique arrow $c \to c'$ and a unique arrow $c' \to c$ (why?). What can you say the compositions of these arrows?

**Example II.7.** In the category $\mathsf{Set}$, the initial object is the empty set and the terminal object is the singleton set.

**Example II.8.** In the categories $\mathsf{Grp}$ and $\mathsf{Ab}$, the trivial group is both initial and terminal.

**Example II.9.** Given a preorder $P$, the terminal object, if it exists, is called the *top object*. The initial object is called the *bottom object*.

The top object is greater than or equal to every other object in the preorder. The bottom object is less than or equal to every other object in the preorder.

Given categories $\mathsf{J}$ and $\mathsf{C}$, we often call a functor $F : \mathsf{J} \to \mathsf{C}$ a *diagram of shape* $\mathsf{J}$ *in* $\mathsf{C}$.

**Definition II.10.** Given two categories $\mathsf{J}$ and $\mathsf{C}$ and an object $c$ in $\mathsf{C}$, let $\underline{c} : \mathsf{J} \to \mathsf{C}$ denote the *constant functor on* $c$ which sends every object in $\mathsf{J}$ to $c$, and every morphism in $\mathsf{J}$ to the identity morphism $\mathrm{id}_c$ on $c$.

**Definition II.11.** Let $\mathsf{J}$ be a small category, and $F : \mathsf{J} \to \mathsf{C}$ be a functor.

A *cone under* $F$ is a pair $(\lambda, c)$, where $c$ is an object in $\mathsf{C}$ and $\lambda$ is a natural transformation $\lambda : F \Rightarrow \underline{c}$. We call $c$ the **nadir** of the cone.

A *cone over* $F$ is a pair $(c, \lambda)$, where $c$ is an object in $\mathsf{C}$ and $\lambda$ is a natural transformation $\lambda : \underline{c} \Rightarrow F$. We call $c$ the **summit** or **apex** of the cone.

Explicitly, the data of a cone $\lambda$ under $F : \mathsf{J} \to \mathsf{C}$ with nadir $c$ is a collection of morphisms $\lambda : F(j) \to c$, indexed by the objects $j$ in $\mathsf{J}$, such that for any morphism $f : j \to k$ in $\mathsf{J}$, the following triangle commutes in $\mathsf{C}$

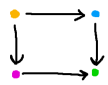
Oftentimes, you will see the word "cocone" instead of "cone under $F$", and in this context usually the word "cone" will refer explicitly to cones over $F$.

$$\begin{tikzcd} F(j) \arrow[rr, "F(f)"] \arrow[dr, "\lambda_j"'] & & F(k) \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

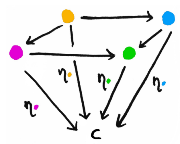
Dually, the data of a cone $\lambda$ over $F : \mathsf{J} \to \mathsf{C}$ with apex $c$ is a collection of morphisms $\lambda_j : c \to F(j)$, indexed by objects $j$ in $\mathsf{J}$, such that for any morphism $f : j \to k$ in $\mathsf{J}$, the following triangle commutes in $\mathsf{C}$

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ F(j) \arrow[rr, "F(f)"] & & F(k) \end{tikzcd}$$

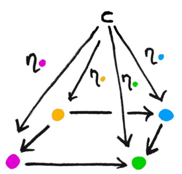
Typically, we think of limits and colimits of functors $F : \mathsf{J} \to \mathsf{C}$ when $\mathsf{J}$ is a relatively "small" or "simple" category. Maybe $\mathsf{J}$ looks something like this



Then if $(c, \eta)$ is a cone under $F$, we have the following image in $\mathsf{C}$:

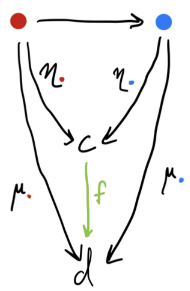


And if $(\eta, c)$ is a cone over $F$, we have the following image in $\mathsf{C}$:



**Definition II.12.** Given a diagram $F$ of shape $\mathsf{J}$ in a category $\mathsf{C}$ (so a functor $F : \mathsf{J} \to \mathsf{C}$) and two cones $(\eta, c)$ and $(\mu, d)$ under $F$, a *morphism of cones under $F$* is a morphism $f \in \mathrm{Mor}(c, d)$ such that for all objects $j$ in $\mathsf{J}$, $\mu_j = f \circ \eta_j$.

Pictorally, a morphism of cones under $F$ connects the nadirs of the cones.



Of course, we have a dual definition for cones over $F$, which connect the apexes of cones. Can you draw a picture?

**Definition II.13.** Given a diagram $F$ of shape J in a category C (so a functor $F : \mathsf{J} \to \mathsf{C}$) and two cones $(c, \eta)$ and $(d, \mu)$ over $F$, a *morphism of cones over* $F$ is a morphism $f \in \mathrm{Mor}(c, d)$ such that for all objects $j$ in J, $\mu_j \circ f = \eta_j$.

**Definition II.14.** Let $F$ be a diagram of shape J in a category C.

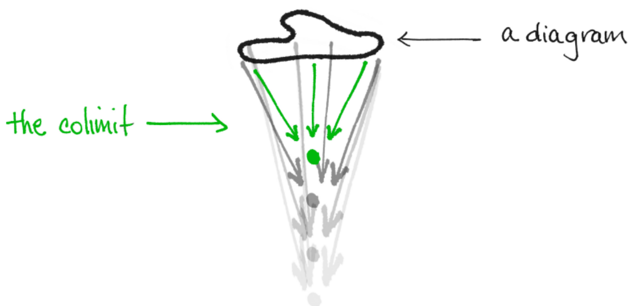
Define $\mathrm{Cone}_{\mathsf{C}}(F)$ to be the category whose objects are cones under $F$, and morphisms are morphisms of cones under $F$.

Conversely, define $\mathrm{Cone}^{\mathsf{C}}(F)$ to be the category whose objects are cones over $F$, and morphisms are morphisms of cones over $F$.

**Definition II.15.** Given a diagram $F$ of shape J in a category C, the *colimit cone* for $F$ is the initial object in $\mathrm{Cone}^{\mathsf{C}}(F)$ (if it exists).

If $(\eta, c)$ is a colimit cone for $F$, then we call the object $c$ the *colimit* of $F$, and denote it by any of the following:

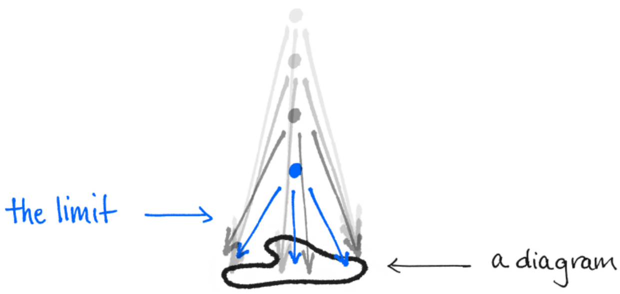
$$\operatorname*{colim}_{\mathsf{J}} F, \qquad \underrightarrow{\operatorname{colim}}\, F, \quad \text{or} \quad \varinjlim F$$




**Definition II.16.** Given a diagram $F$ of shape J in a category C, the *limit cone* for $F$ is the terminal object in $\mathrm{Cone}^{\mathsf{C}}(F)$ (if it exists).

If $(c, \eta)$ is a limit cone for $F$, then we call the object $c$ the *limit* of $F$, and denote it by any of the following:

$$\lim_{\mathsf{J}} F \quad \text{or} \quad \varprojlim F$$




## II.3 Examples of limits and colimits

I'll take a moment to say something about category theory: the above definitions were quite technical and abstract. Like most definitions in category theory, the definition of the (co)limit cannot be internalized or understood by reading it. You need to work through examples, preferably as many as possible. Thankfully, you already know lots of examples of limits and colimits!

I am going to explicitly give an example of computing a colimit and a limit in Set.

**Example II.17.** Let J be a category with $n > 0$ objects and no non-identity morphisms. Then a functor $F : \mathsf{J} \to \mathsf{C}$ is the data of a choice of $n$ objects $X_1, \ldots, X_n$ in C. Then the (co)limit of $F$ is called the **(co)product** of the $X_i$'s.

In Set, coproducts are disjoint unions and products are cartesian products of sets.

In Ab, the coproduct is given by direct sum, and the product is given by the product of groups.

In Top, the coproduct is given by the disjoint union of spaces with the disjoint union topology, while the product is given by the cartesian product with the product topology.

In $\mathsf{Top}_*$, the coproduct is given by the wedge product (join spaces at their basepoint), while the product is the regular product in Top.

**Example II.18.** An **equalizer** is a limit of a diagram indexed by the **parallel pair**, the category $\bullet \rightrightarrows \bullet$ with two objects and two parallel non-identity morphisms between them.

**Example II.19** (Limits in Set)**.** In general, limits in Set can be described completely as follows:

Given a diagram $S : \mathsf{J} \to \mathsf{Set}$, define the set

$$\varprojlim S := \{(s_i)_i \in \prod_{\mathsf{J}} Si : \forall \phi : i \to i', (S\phi)(s_i) = s_{i'}\}.$$

One can check that this is a limit of $F$.

**Example II.20** (Colimits in Set)**.** Given a diagram $S : \mathsf{J} \to \mathsf{Set}$, define the set

$$\operatorname{colim}_{\longrightarrow} S := \left( \coprod_{\mathsf{J}} Si \right) / (s_i \sim s_{i'} \text{ if } \exists \phi : (S\phi)(s_i) = s_{i'}).$$

One can check that this is a colimit of $F$.

**Example II.21** ((Co)limits in Top)**.** Like Set, the category Top is also complete and cocomplete. A limit in Top is formed by taking the limit of underlying sets and endowing it with the subspace topology. Likewise, a colimit in Top is formed by taking the colimit of underlying sets and endowing it with the quotient topology.

## II.4 The $\mathrm{Hom}$ functor

Let $X, Y$ be objects in C. We consider the set

$$\mathrm{Hom}_{\mathsf{C}}(X, Y)$$

of all morphisms $X \to Y$ in C. (When C is understood, we just write $\mathrm{Hom}(X, Y)$.) Given a morphism $f : Y \to Y'$, post-composition defines a function $f \circ - : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y')$. Given a morphism $g : X \to X'$, pre-composition defines a function $- \circ g : \mathrm{Hom}(X', Y) \to \mathrm{Hom}(X, Y)$. Notice that this map goes the “other way."

This describes two important functors.

**Definition II.22.** Let C be a category. For each $c \in \mathsf{C}$, we define the *covariant* $\mathrm{Hom}$ *functor*

$$\mathrm{Hom}_{\mathsf{C}}(c, -) : \mathsf{C} \to \mathsf{Set}.$$

**Definition II.23.** Let C be a category. For each $c \in \mathsf{C}$, we define the *contravariant* $\mathrm{Hom}$ *functor*

$$\mathrm{Hom}_{\mathsf{C}}(-, c) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}.$$

**Exercise II.24.** Think about the contravariant Hom functor.

**Exercise II.25.** In a category C, prove that a morphism $f : X \to Y$ is an isomorphism $\iff$ for every object $Z \in \mathsf{C}$, the morphism $\mathrm{Hom}(Z, f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is an isomorphism. (I.e., iff the function $f \circ -$ is a bijection.)

(Not totally essential part.) The two Hom functors fit together in such a way that we can turn the co/contravariant Hom functors into a single Hom functor. We mean the following.

**Proposition II.26.** *If $f : X \to Y$, $g : X' \to Y'$ are morphisms in C, then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(X, Y) \arrow[r, "\mathrm{Hom}(X,g)"] \arrow[d, "\mathrm{Hom}(f,Y)"'] & \mathrm{Hom}_{\mathsf{C}}(X, Y') \arrow[d, "\mathrm{Hom}(f,Y')"] \\
\mathrm{Hom}_{\mathsf{C}}(X', Y) \arrow[r, "\mathrm{Hom}(X',g)"] & \mathrm{Hom}_{\mathsf{C}}(X', Y')
\end{tikzcd}$$

*Thus, we have a functor* $\mathrm{Hom}_{\mathsf{C}}(-, -) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$.

## II.5 Adjunctions

There is a notion of adjunctions. The slogan is, “The slogan is, *adjunctions are everywhere.*" There are several equivalent definitions. None is “the best," they are very much so all useful. The idea is that two functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ may “undo each other on the level of hom sets;" we make this precise.

**Definition II.27.** A *hom-set adjunction* is a pair of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ together with a natural isomorphism

$$\Phi : \mathrm{Hom}_{\mathsf{D}}(F-,-) \cong \mathrm{Hom}_{\mathsf{C}}(-,G-).$$

**Definition II.28.** A *unit-counit adjunction* is a pair of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ together with natural transformations $\eta : \mathrm{id}_{\mathsf{C}} \implies GF$ and $\epsilon : FG \implies \mathrm{id}_{\mathsf{D}}$ such that the following diagrams commute (we call these the *triangle identities*).

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

**Proposition II.29.** *Let $(F, G, \Phi)$ be a hom-set adjunction. Define its canonical unit-counit structure as follows.*

- *For $c \in \mathsf{C}$, we define the morphism $\eta_c := \Phi_{c,Fc}(\mathrm{id}_{Fc})$; and*
- *For $d \in \mathsf{D}$, we define the morphism $\epsilon_d := \Phi^{-1}_{Gd,d}(\mathrm{id}_{Gd})$.*

*The claim is that $(F, G, \eta, \epsilon)$ is a unit-counit adjunction, i.e. the $\eta_c, \epsilon_d$'s assemble to natural transformations satisfying the triangle identities.*

**Proposition II.30.** *Let $(F, G, \eta, \epsilon)$ be a unit-counit adjunction. Define its canonical hom-set adjunction structure as follows. For each $c \in \mathsf{C}$, $d \in \mathsf{D}$, and $f \in \mathrm{Hom}_{\mathsf{D}}(Fc, d)$, define $\Phi(f) := \eta_c \circ Gf \in \mathrm{Hom}_{\mathsf{C}}(c, Gd)$. The claim is that $(F, G, \Phi)$ is a hom-set adjunction.*

The last two propositions say that every hom-set adjunction gives rise to a unit-counit adjunction and vice-versa.

**Proposition II.31.** *Hom-set adjunctions are “the same thing as" unit-counit adjunctions.*

**Example II.32** (Cool example from Peter's talk)**.** We work in $\mathsf{Top}^*$, the category of based spaces.[3] Given based spaces $X, Y$, we may ask for some based space $G(X, Y)$ such that for every $Z$,

$$\mathrm{Hom}(G(X,Y), Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)). \tag{33}$$

In this sense, $G(X, Y)$ “undoes" the functor “take the space of based maps out of $Y$" on the level of hom-sets. In fact, such a based space $G(X, Y)$ exists for every $X$ and $Y$, and the isomorphisms (32) are such that they describe an adjunction. (In the language above, a *hom-set adjunction*.) The space $G(X, Y)$ is the *smash product of based spaces*, defined as

$$G(X, Y) = X \wedge Y := (X \times Y)/(X \vee Y).$$

Precisely, the functors $- \wedge Y : \mathsf{Top}^* \to \mathsf{Top}^*$ and $\mathrm{Hom}(Y, -) : \mathsf{Top}^* \to \mathsf{Top}^*$ are adjoint. Smashing with $Y$ is left adjoint to homming out of $Y$.

This matters because if we take $Y = S^1$, our adjoint functors specialize to important constructions: given a space $X$, one has

$$Y \wedge X = \Sigma X \quad \text{and} \quad \mathrm{Hom}(Y, X) = \Omega X.$$

The spaces $\Sigma X, \Omega X$ are called the *suspension* and *loop space* of $X$, respectively. They are essential to doing algebraic topology and homotopy theory. Then, being instances of functors which are adjoint, the adjunction tells us that for any space $W$ we get

$$\mathrm{Hom}(\Sigma X, W) \cong \mathrm{Hom}(X, \Omega W)$$

and these isomorphisms are *natural*. I could go on about why this is great.

[3]Objects are spaces with a chosen point $(X, x_0)$ and morphisms $(X, x_0) \to (Y, y_0)$ are continuous maps $f : X \to Y$ such that $f(x_0) = y_0$.

## II.6 More $\mathrm{Hom}$ functor

Are hom-sets between (co)limits the (co)limits of hom-sets? The important answer is yes, and we will be thinking about questions like this more later.

**Proposition II.34.** *For every object $X$, the functors $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$ commute with limits. That is, if $\mathsf{I} \to \mathsf{C}$ is a diagram in $\mathsf{C}$ and $\varprojlim_{\mathsf{I}} c_i$ exists, then for any object $X$ one has*

$$\mathrm{Hom}_{\mathsf{C}}(X, \varprojlim_{\mathsf{I}} c_i) \cong \varprojlim_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(X, c_i) \qquad \textit{and}$$

$$\mathrm{Hom}_{\mathsf{C}}(\varprojlim_{\mathsf{I}} c_i, X) \cong \mathrm{co}\varinjlim_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(c_i, X).$$

**Proposition II.35.** *For every object $X$, the functors $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$ commute with colimits. (The precise statement of this is dual to that in the previous proposition.)*

# III (Review) Lecture 3: Sheaves on spaces

## III.1 What is a sheaf?

Let $X$ be a topological space.

**Definition III.1.** We let $\mathrm{Open}(X)$ be the category whose objects are open subsets $U \subseteq X$, and where morphisms $U \to V$ are precisely inclusions $U \subseteq V$.

In particular, between any two objects $U$ and $V$, there are either exactly 0 maps between them (if $U$ does not contain $V$), or exactly 1 map $U \to V$ (if $U \subseteq V$).

As a category, this is not so interesting. It is introduced only to make the following notions easier to define.

**Definition III.2** (Presheaves)**.** A *presheaf* on a topological space $X$ is a contravariant functor

$$F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathsf{Ab},$$

for $\mathsf{Ab}$ the category of abelian groups.

**Remark III.3.** You can replace $\mathsf{Ab}$ by any category you like, and get a different notion of presheaf; we will focus primarily on presheaves valued in abelian groups today, but other common use cases are presheaves valued in the category of sets, or presheaves valued in the category of rings.

**Example III.4.** As one example of a presheaf, we define the *sheaf of continuous functions on a space* $X$ (this presheaf, as the name suggests, will turn out to also be a sheaf).

It is the presheaf $\mathcal{F}$ on $X$ given by

$$\mathcal{F}(U) := \{\text{continuous functions } \phi : U \to \mathbb{R}\}.$$

If $U \subseteq V$, then the map $\mathcal{F}(V) \to \mathcal{F}(U)$ (recall that, in addition to specifying abelian groups $\mathcal{F}(U)$, we also need to specify where our functor maps morphisms!) is defined to be the restriction map

$$\phi \mapsto \phi|_U,$$

where for $\phi : V \to \mathbb{R}$, the function $\phi|_U$ is just its restriction to $U$.

For a general presheaf $\mathcal{F}$, we often call the map $\mathcal{F}(V) \to \mathcal{F}(U)$ the *restriction* from $V$ to $U$, and we call elements of $\mathcal{F}(U)$ *sections of $\mathcal{F}$ defined over $U$*. We often abbreviate the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ by just writing $s|_U$, where $s \in \mathcal{F}(V)$.

The definition of sheaf is intended to capture two very important general properties of the sheaf of continuous functions.

**Definition III.5.** A *sheaf* on a space $X$ is a presheaf $\mathcal{F}$ on $X$ so that for any open subset $U$ of $X$, any open cover $\{U_i\}_{i \in I}$ of $U_i$, and any collection of $s_i \in \mathcal{F}(U_i)$ so that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

there exists a unique $s \in \mathcal{F}(U)$ so that, for each $i$,

$$s|_{U_i} = s_i.$$

**Remark III.6.** There are two parts to this notion: *existence* of glueings, and *uniqueness* of glueings.

## III.2 The categories of sheaves and presheaves

**Definition III.7.** A *morphism of presheaves* is just a natural transformation of functors. A *morphism of sheaves* is just a morphism of the underlying presheaves.

We write $\mathsf{Psh}(X), \mathsf{Sh}(X)$ to denote the categories of presheaves and sheaves on $X$.

There is a *fully faithful* forgetful functor

$$I : \mathsf{Sh}(X) \to \mathsf{Psh}(X),$$

since every sheaf is a presheaf.

There are some useful properties of the category of sheaves and presheaves.

**Theorem III.8.** *The category* $\mathsf{Psh}(X)$ *of presheaves on* $X$ *is complete and cocomplete.*

*In fact, if* $\mathcal{F}_i$ *is a diagram of presheaves, then* $\varprojlim_i \mathcal{F}_i$ *is the presheaf*

$$\left(\varprojlim_i \mathcal{F}_i\right)(U) := \varprojlim_i \mathcal{F}_i(U),$$

*where the limit on the right is taken in the category of abelian groups. A similar formula holds for colimits.*

*Proof.* Left to the reader; as a hint, construct a map

$$\mathrm{Hom}(\mathcal{G}, \varprojlim_i \mathcal{F}_i) \to \varprojlim_i \mathrm{Hom}(\mathcal{G}, \mathcal{F}_i)$$

(for $\varprojlim_i \mathcal{F}_i$ the sheaf defined in the theorem statement) and check it is both injective and surjective; this proves our formula is actually the categorical limit. □

Our next step will be to prove that $\mathsf{Sh}(X)$ is complete and cocomplete. First, we show completeness.

**Theorem III.9.** *Let* $\mathcal{F}_i$ *be a diagram of* sheaves. *Then the presheaf limit*

$$\left(\varprojlim_i \mathcal{F}_i\right)(U) := \varprojlim_i \mathcal{F}_i(U)$$

*is a sheaf.*

*Proof.* For this, it is useful to give a reformulation of the definition of sheaf. Let $\mathcal{F}$ be a presheaf. Then $\mathcal{F}$ is a sheaf if and only if, for any open set $U \subseteq X$, and any open cover $\{U_\alpha\}_{\alpha\in A}$ of $U$, the sequence of maps

$$0 \to \mathcal{F}(U) \to \prod_{\alpha\in A} \mathcal{F}(U_\alpha) \to \prod_{\alpha,\beta\in A} \mathcal{F}(U_\alpha \cap U_\beta)$$

is left exact, where the first map is just the product of the restriction maps, and the second map sends $(s_\alpha)_{\alpha\in A}$ to

$$(s_\alpha|_{U_\alpha\cap U_\beta} - s_\beta|_{U_\alpha\cap U_\beta})_{\alpha,\beta\in A}.$$

Or equivalently, $\mathcal{F}$ is a sheaf if and only if the map

$$\mathcal{F}(U) \to \ker\left(\prod_{\alpha\in A} \mathcal{F}(U_\alpha) \to \prod_{\alpha,\beta\in A} \mathcal{F}(U_\alpha \cap U_\beta)\right)$$

is always an isomorphism.

Since each $\mathcal{F}_i$ is a sheaf, the maps

$$\mathcal{F}_i(U) \to \ker\left(\prod_{\alpha\in A} \mathcal{F}_i(U_\alpha) \to \prod_{\alpha,\beta\in A} \mathcal{F}_i(U_\alpha \cap U_\beta)\right)$$

are all isomorphisms. Take the limit of these diagrams over $i$, and so the map

$$\varprojlim_i \mathcal{F}_i(U) \to \varprojlim_i \ker\left(\prod_{\alpha\in A} \mathcal{F}_i(U_\alpha) \to \prod_{\alpha,\beta\in A} \mathcal{F}_i(U_\alpha \cap U_\beta)\right)$$

is an isomorphism. Limits commute with limits; the kernel is a limit, and so it commutes with $\varprojlim_i$. It also commutes with all the products, and hence the above isomorphism can be rewritten as an isomorphism

$$\varprojlim_i \mathcal{F}_i(U) \to \ker\left(\prod_{\alpha\in A} \varprojlim_i \mathcal{F}_i(U_\alpha) \to \prod_{\alpha,\beta\in A} \varprojlim_i \mathcal{F}_i(U_\alpha \cap U_\beta)\right).$$

Thus $\varprojlim_i \mathcal{F}_i$ is a sheaf. □

**Corollary III.10.** *The category* $\mathsf{Sh}(X)$ *is complete.*

Cocompleteness is harder. We state, without proof, a useful theorem.

**Theorem III.11** (Sheafification)**.** *The fully faithful forgetful functor*

$$I : \mathsf{Sh}(X) \to \mathsf{Psh}(X)$$

*has a left adjoint functor*

$$\# : (X) \to \mathsf{Sh}(X),$$

*called* sheafification*.*

Since $I$ is fully faithful, this adjunction automatically obeys the property that the morphism

$$\mathcal{F} \to \mathcal{F}^{\#}$$

from a *sheaf* $\mathcal{F}$ into its sheafification is an isomorphism.

**Theorem III.12.** *The category* $\mathsf{Sh}(X)$ *is cocomplete.*

*Proof.* Let $\mathcal{F}_i$ be a diagram of sheaves, and let $\mathcal{G}$ be the *presheaf* colimit. Then we claim that $\mathcal{G}^{\#}$ is the colimit of the $\mathcal{F}_i$ in the category of sheaves. Indeed, for any sheaf $\mathcal{F}'$,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathcal{G}^{\#}, \mathcal{F}') &= \mathrm{Hom}_{\mathsf{Psh}(X)}(\mathcal{G}, \mathcal{F}') \\ &= \varprojlim_i \mathrm{Hom}_{\mathsf{Psh}(X)}(\mathcal{F}_i, \mathcal{F}') \\ &= \varprojlim_i \mathrm{Hom}_{\mathsf{Sh}(X)}(\mathcal{F}_i, \mathcal{F}'),\end{aligned}$$

and so $\mathcal{G}^{\#}$ is the colimit. □

**Warning III.13.** The colimit of a family of presheaves is *not* computed as

$$\left(\operatorname*{colim}_{\overrightarrow{i}} \mathcal{F}_i\right)(U) := \operatorname*{colim}_{\overrightarrow{i}} \mathcal{F}_i(U),$$

in contrast to the case of limits. You have to use a sheafification procedure.

## III.3 The inverse and direct image functors

**Definition III.14.** Let $f : X \to Y$ be a continuous function of topological spaces $X, Y$. Then we define the *direct image functor*

$$f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$$

and the *inverse image functor*

$$f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$$

as follows. The functor $f_*$ sends a sheaf $\mathcal{F}$ on $X$ to

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)),$$

and $f^{-1}$ sends a sheaf $\mathcal{G}$ on $Y$ to the *sheafification* of the presheaf

$$(f^{-1}\mathcal{G})(U) := \mathrm{co}\varinjlim_{V \subseteq f(U)} \mathcal{G}(V).$$

**Theorem III.15.** *The functors $f^{-1}, f_*$ are an adjoint pair. Thus, for any $\mathcal{F} \in \mathsf{Sh}(X), \mathcal{G} \in \mathsf{Sh}(Y)$, we have*

$$\mathrm{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathsf{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

*In particular, $f^{-1}$ is right exact and $f_*$ is left exact.*

*Proof.* Left to readers for time. □

We now look at some particularly important examples of this construction.

## III.4 Sections and constant sheaves

For any topological space $X$, there is a unique continuous map $f : X \to \{*\}$, for $\{*\}$ denoting the one point topological space.

The category $\mathsf{Sh}(\{*\})$ is equivalent to to $\mathsf{Ab}$, since a sheaf on a point is just the value of its global sections on the entire space. Thus $f^{-1}, f_*$ give us functors

$$f^{-1} : \mathsf{Ab} \to \mathsf{Sh}(X)$$

and

$$f_* : \mathsf{Sh}(X) \to \mathsf{Ab}.$$

The direct image is simple:

$$(f_*\mathcal{F})(\{*\}) = \mathcal{F}(f^{-1}(\{*\})) = \mathcal{F}(X).$$

It has a special name and notation.

**Definition III.16.** The *global sections functor* is the functor $\Gamma_X : \mathsf{Sh}(X) \to \mathsf{Ab}$ given by $\Gamma_X(\mathcal{F}) = \mathcal{F}(X)$.

The global sections functor has an adjoint $f^{-1}$, which is called the constant sheaf functor.

**Definition III.17.** Let $A$ be an abelian group. Then *the constant sheaf on $A$* is the sheaf $\underline{A} := f^{-1}(A)$.

**Remark III.18.** As an exercise, prove that $\underline{A}(U) = A$ if $U$ is connected. What happens if $U$ is disconnected?

**Remark III.19.** Since the global sections functor is a right adjoint, it commutes with all limits. This gives a category theoretic explanation for why the formula

$$\left(\varprojlim_i \mathcal{F}_i\right)(X) = \mathcal{F}_i(X)$$

holds at the level of sheaves.

The global sections functor in general does not commute with colimits, though, which is why it is difficult to describe the colimit of a diagram of sheaves directly.

## III.5 Restriction and extension

Let $X$ be a topological space, $U$ an open subset, and $j : U \to X$ the inclusion.

**Definition III.20.** The functor $j^{-1} : \mathsf{Sh}(X) \to \mathsf{Sh}(U)$ is *restriction to $U$*. Sometimes $j^{-1}\mathcal{F}$ is written $\mathcal{F}|_U$.

It is easy to verify that

$$(j^{-1}\mathcal{F})(V) = \mathcal{F}(V).$$

## III.6 Skyscraper sheaves and stalks

For any point $x \in X$, there is a continuous inclusion $i_x : \{x\} \to X$. This gives us two functors

$$i_{x,*} : \mathsf{Ab} \to \mathsf{Sh}(X),$$

$$i_x^{-1} : \mathsf{Sh}(X) \to \mathsf{Ab}.$$

**Definition III.21.** Let $\mathcal{F}$ be a sheaf on a space $X$. The *stalk of $\mathcal{F}$ at $x$* is the abelian group

$$\mathcal{F}_x = i_x^{-1}(\mathcal{F}) = \operatorname{co}\varinjlim_{U \ni x} \mathcal{F}(U).$$

The stalk is an incredibly useful construction. Here are some properties one should verify.

**Definition III.22.**

Check that the colimit definition of a stalk makes sense for presheaves, so that it makes sense to talk about stalks.

A presheaf $\mathcal{F}$ and its sheafification $\mathcal{F}^{\#}$ have the same stalk at every point $x$. If you know the construction of the sheafification functor, you can check this directly – but you don't need to know the construction of sheafification! Try to prove this purely using categorical properties of stalks and sheafification.

If $U \subseteq X$ is open and $x \in U$, then there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$. If $s \in \mathcal{F}(U)$, we write $s_x$ for the image of $s$ under this natural map. (Hint: remember the definition of colimit!)

Let $\mathcal{F}$ be a presheaf. Then the axiom of uniqueness of glueings in the definition of sheaf is equivalent to saying that, for every section $s \in \mathcal{F}(U)$, if $s_x = 0$ for every $x \in U$, then $s = 0$.

Taking stalks commutes with limits and colimits.

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a sequence of sheaves, then it is short exact if and only if for every $x \in X$, the sequence $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ is.

Use the previous point to prove that sheafification is exact.

**Definition III.23.** The *skyscraper sheaf* at a point $x \in X$ with stalk $A$ is the sheaf $i_{x,*}A$.

# IV (Review) Lecture 4: Sheaves on sites

In 1949, Andre Weil (while he was a UChicago professor) proposed the Weil conjectures, which propose a way of taking a system of equations defined over $\mathbb{Z}$, and relating the number of solutions to that system over a finite field $\mathbb{F}_q$ to the topology of the complex solution set. Weil was able to prove the Weil conjectures in a few special cases; most notably, Weil algebraized a large amount of the modern theory of algebraic geometry of curves, so that classical theorems for curves over $\mathbb{R}$ or $\mathbb{C}$ would apply to fields like $\mathbb{F}_q$, allowing him to give a proof of the Weil conjectures for curves.

As Weil proposed from the start, and as Serre made more precise in 1960 with his proof of the Kahler analogues of the Weil conjectures, the ultimate difficulty in proving the Weil conjectures stems from being able to define "Weil cohomology theories" – basically, a theory of cohomology for algebraic varieties defined over fields like $\mathbb{F}_q$.

Ultimately, Grothendieck defined etale cohomology to fulfill this dream.

## IV.1 Sites

The central difficulty with giving a cohomology theory on algebraic varieties over finite fields is that a finite topological space doesn't have that many open sets, and so if you just take an algebraic topologist's cohomology then you won't get anything useful.

Grothendieck resolved this by extending the definition of cohomology. We won't talk about this generalized cohomology (yet! but later on in condensed math, we will need to), but the starting point is a generalization of topological space, and a corresponding generalization of the notion of sheaf.

**Definition IV.1.** Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of distinguished families of morphisms sharing a common target, called *covering families* (so, a covering family is the data of an object $U \in \mathcal{C}$, and a collection of morphisms $\{U_i \to U\}_{i\in I}$), obeying the following axioms, analogous to the axioms for a topology.

Firstly, any isomorphism is always a covering family. This is in analogue to how the entire set of a topological space is open.

Secondly, analogously to how an arbitrary union of opens is open, if $\{U_i \to U\}_{i\in I}$ is a covering family, and if for each $i \in I$ we have a covering family $\{U_{j_i} \to U_i\}_{j_i \in J_i}$, then the family

$$\{U_{j_i} \to U\}_{i\in I, j_i \in J_i}$$

is a covering family of $U$.

Lastly, analogously to how finite intersections of open sets are open, we need one more axiom. Firstly, the categorical analogue of the intersection is something called the *fiber product*; if the reader doesn't know what it is, then it is given in the definition just below this one.

So, we require that $\mathcal{C}$ has all fiber products, and that if $V \to U$ is any morphism, and $\{U_i \to U\}_{i\in I}$ is any covering family, then $\{U_i \times_U V \to V\}_{i\in I}$ is a covering family as well.

**Definition IV.2.** If $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ are two morphisms, then the *fiber product* of $f_1, f_2$ is the limit $X_1 \times_S X_2$ of the diagram

$$\begin{tikzcd} X_1 \times_S X_2 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & S \end{tikzcd}$$

## IV.2 Sheaves on a site

**Definition IV.3.** Let $\mathcal{C}$ be a site (that is, a category with a Grothendieck topology). Then the category $\mathsf{Psh}(\mathcal{C})$ of *presheaves* on $\mathcal{C}$ is the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$.

The category $\mathsf{Sh}(\mathcal{C})$ of *sheaves* on $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of presheaves $\mathcal{F}$ to that, for every covering family $\{U_i \to U\}_{i\in I}$, $\mathcal{F}(U)$ is the equalizer of the two morphisms

$$\prod_{i\in I} \mathcal{F}(U_i) \to \prod_{i,j} \in \mathcal{F}(U_i \times_U U_j).$$

(The two morphisms are

$$(s_i)_{i\in I} \mapsto (s_i|_{U_i\times_U U_j})_{i,j\in I},$$
$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\times_U U_j})_{i,j\in I}$$

obtained via the two natural maps $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$.)

The results of sheaf theory we established last time essentially go through for the exact same abstract nonsense categorical reasons.

## IV.3 Weil conjectures

Instead of reproving the same theorems for sites, I want to spend some time talking about why they were originally introduced: to solve the Weil conjectures.

What are the Weil conjectures? I don't want to be too precise, but basically they are a series of conjectures of the following form (although, the actual ones are a little more general than what I am about to say).

Imagine we have a system of polynomial equations in some number of variables, where the coefficients in the equations are all algebraic numbers, so something like

$$y - x^2 = \sqrt{2}.$$

This system of equations defines a complex algebraic variety, which has (singular) cohomology groups, a very concrete topological invariant.

But over certain finite fields, the equation

$$y - x^2 = \sqrt{2}$$

still makes sense (for instance, for $p$ odd, $\sqrt{2}$ exists over either $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$, and so this equation still makes sense).

Over a finite field, there's only some finite number of solutions to this equation. You could take singular cohomology of this finite set (with its Zariski topology), but the cohomology groups you get are essentially meaningless.

Is there any sense in which the finite field remembers the cohomology of the complex variety?

Weil found a miracle: it does! I want to look at an example which will be very easy to count points for, but which might not be so convincing. There was much more numerical evidence known to Weil (including a few special cases of the Weil conjectures that Weil proved before he even formulated the Weil conjectures, and the case of elliptic curves which was proven by Hasse in 1933).

Look at $\mathbb{P}^n_{\mathbb{F}_q}$, the projective space over $\mathbb{F}_q$. Recall that the cohomology of projective $n$-space over the complex numbers is given

$$H^*(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}[\alpha]$$

where $|\alpha| = 2$. In other words, in even degrees, $H^{2i}(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}$ for $0 \le i \le n$, and $H^{2i-1}(\mathbb{P}^n_{\mathbb{C}}) = 0$ in all odd degrees. There are many ways to compute this, but probably the easiest is to note that $\mathbb{P}^n_{\mathbb{C}}$ can be built up as a CW complex built out of a one 0-cell, one 2-cell, ..., one $2n$-cell. (Do the example of a disk glued along its boundary is a sphere.)

How many points are on $\mathbb{P}^n_{\mathbb{F}_q}$? This is easy: it's just

$$1 + q + \cdots + q^n.$$

Look at the analogy between how $\mathbb{P}^n_{\mathbb{C}}$ is built out of one 0-cell, one 2-cell, etc.

The Weil conjectures predict that in general, the number of points defined over $\mathbb{F}_q$ should be intimately related to the cohomology groups of the corresponding complex algebraic variety. Specifically, Weil defines the *zeta function* of a variety $X$ over $\mathbb{F}_q$ as a certain exponential generating function of the sequence $|X(\mathbb{F}_q)|, |X(\mathbb{F}_{q^2})|, \ldots$, and then conjectures that this generating function $\zeta$ should obey certain properties related to the topology of the corresponding complex variety.

As another example, we consider the case of elliptic curves. An elliptic curve $X$ recall is a certain type of equation of the form

$$y^2 = x^3 + ax + b.$$

Over the complex numbers, such equations correspond to genus 1 surfaces.

Over a finite field $\mathbb{F}_q$, Hasse proved that the number of points on an elliptic curve was of the form

$$|\#X(\mathbb{F}_q) - (q+1)| \leq 2\sqrt{q}.$$

How does this relate to topology? Well, over the complex numbers,

$$H^0(X;\mathbb{Z}) = \mathbb{Z},$$

$$H^1(X;\mathbb{Z}) = \mathbb{Z}^2.$$

$$H^2(X;\mathbb{Z}) = \mathbb{Z},$$

The $\mathbb{Z}, \mathbb{Z}$ in the even terms give us the $1+q$, just like in the projective space case the even cohomology gave us $1 + q + \cdots + q^n$. The $\mathbb{Z}^2$ in the odd degree terms give us that

$$\#X(\mathbb{F}_q) = 1 + q + \alpha + \overline{\alpha},$$

where $\alpha$ is a complex number of absolute value $\sqrt{q}$. It's easy to deduce the Hasse bound from this.

## IV.4 Etale cohomology

So, it seems that varieties over finite fields can see shadows of the cohomology of the corresponding complex variety. Can one make this precise?

It turns out yes, via etale cohomology. Grothendieck had the insight that the Zariski topology on a variety over a finite field failed to give enough cohomology because it failed to have enough open sets. Grothendieck replaced it with the etale topology.

Basically, an etale morphism of varieties is the algebraic geometer's analogue of a 'local diffeomorphism' in differential topology. Some examples of etale maps are covering maps and open inclusions, for instance.

Grothendieck defined the (small) etale site over a base scheme $S$ to be the category of all etale morphisms of schemes $X \to S$ (morphisms being commuting triangles), endowed with the etale topology: an open cover in this category is then just any collection of morphisms $\{\phi_i : X_i \to X\}$ which are jointly surjective.

Topoi give rise to notions of sheaves and cohomology theories. One of the really cool useful results is the following comparison theorem: if $X$ is a smooth complex variety, then for any *finite* ring $\Lambda$,

$$H^i_{\text{et}}(X, \underline{\Lambda}) \cong H^i(X;\Lambda),$$

where the left hand side is etale cohomology with coefficients in the constant sheaf associated to $\Lambda$, and the right hand side is singular cohomology.

So, the etale cohomology remembers the complex cohomoloy, at least for finite coefficient rings! One can proceed to define the $\ell$-adic cohomology of $X$ to be

$$H^i_{\text{et}}(X, \mathbb{Z}_\ell) := \varprojlim_n H^i_{\text{et}}(X, \mathbb{Z}/\ell^n\mathbb{Z}).$$

We remark that in general, the etale cohomology in the constant sheaf $\mathbb{Z}_\ell$ is NOT equal to the $\ell$-adic cohomology we defined above. Very sad.

It would be nice if we could have etale cohomology with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$, but it seems like the $\ell$-adic ones are the best we can do.

# V Lecture 5: Condensed mathematics?

The "review lectures" are finished. Now, we are going to make a serious move on condensed math. Although we still are not getting to "real condensed stuff" until next week. But let's finally ask "why condensed math?"

Here are some very basic answers.

(1) A lesson going back to Grothendieck: **we prefer good categories with complicated objects over bad categories with familiar objects.** Bad categories sometimes arise when we mix topology and algebra. For instance, the category TopAb of topological abelian groups is *not abelian*. Consider the following morphism in TopAb:

$$(\mathbb{R}, \text{discrete}) \hookrightarrow (\mathbb{R}, \text{Euclidean}).$$

This map has trivial kernel and cokernel, but is *not* an isomorphism. *Condensed abelian groups* fix this problem. (We will work out this example shortly.)

(2) More broadly, the category Top is not particularly well-behaved. It does not have an internal $\mathrm{Hom}$ is general, nor does it have any "obvious" subcategory of generators. *Condensed sets* fix this problem.

(3) Scholze, here: "*Topological spaces formalize the idea of spaces with a notion of "nearness" of points. However, they fail to handle the idea of "points that are infinitely near, but distinct" in a useful way. Condensed sets handle this idea in a useful way.*" For the most basic example, consider the action of $\mathbb{Q}$ on $\mathbb{R}$. The quotient space $\mathbb{R}/\mathbb{Q}$ has infinitely many points, but the quotient topology is indiscrete—no two points can be separated. We have lost topological information. More important examples like this arise in algebraic geometry, functional analysis, ...

(4) Another good reason is because Peter Scholze says that developing condensed math might be one of the most important things he will ever do.

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## V.1 Naive condensed sets

The proposal is to replace topological spaces with condensed sets. These will be sheaves on the site of (compact, Hausdorff) spaces.

**Definition V.1.** Denote by cHaus the category of compact Hausdorff spaces. The collection of "finite, jointly-surjective maps" forms a Grothendieck topology for cHaus. Precisely, a family of morphisms $\{U_i \to U\}_I$ is in this Grothendieck topology iff $I$ is finite and each $u \in U$ is hit by some $U_i \to U$.

**Definition V.2.** A *naive condensed set* is a sheaf of sets on the site cHaus. In other words, a functor $F : \mathsf{cHaus}^{op} \to \mathsf{Set}$ such that

(1) If $A, B \in \mathsf{cHaus}$, then the natural map $F(A \amalg B) \to F(A) \times F(B)$ is a bijection.

(2) If $A \to B$ is a surjection in cHaus, then the natural map $F(B) \to \{a \in F(A) : p_1^*(a) = p_2^*(a) \in F(A \times_B A)\}$ is a bijection. (In other words, that map realizes $F(B)$ as an equalizer of the two projections $F(A) \to F(A \times_B A)$.)

**Remark V.3.** This definition has set-theoretic issues. The essential quirk is that the category cHaus is large: there is more than a set's worth of compact Hausdorff spaces. In general, strange things happen with large categories, in particular there is no "category of functors $\mathsf{C} \to \mathsf{D}$" if $\mathsf{C}$ is large. We will spend most of this seminar correcting this issue. In doing so, we will learn foundational condensed theory and get some categorical exercise. No actual set theory will be involved (for those worried). We will get to more interesting theory toward the end(?)

**Example V.4.** Let's ignore set-theoretic issues for a moment. Recall the example from (1) above: *in* TopAb*, the morphism $i : (\mathbb{R}, \text{discrete}) \to (\mathbb{R}, \text{Euclidean})$ has trivial cokernel and kernel, thus should be an isomorphism, but is not.* Condensed theory says to replace TopAb with the category of *condensed* *abelian groups*, i.e. with *sheaves of abelian groups* on $\mathsf{cHaus}$. Every space gives rise to a condensed set via $X \mapsto \underline{X} := \mathrm{Hom}_{\mathsf{Top}}(-, X)$, and the morphism $i : (\mathbb{R}, \text{discrete}) \to (\mathbb{R}, \text{Euclidean})$ induces a morphism of condensed abelian groups

$$i : \underline{(\mathbb{R}, \text{discrete})} \to \underline{(\mathbb{R}, \text{Euclidean})}.$$

We can compute the kernel and cokernel of this map! The kernel is trivial, however the cokernel is the sheaf

$$\mathsf{cHaus}^{\mathrm{op}} \ni S \mapsto \{\text{continuous maps } S \to (\mathbb{R}, \text{Euclidean})\}/\{\text{continuous maps } S \to (\mathbb{R}, \text{discrete})\} \in \mathsf{Ab}.$$

This is not trivial in general. For instance, consider $S = (\mathbb{R}, \text{Euclidean})$. The function $f : S \to (\mathbb{R}, \text{Euclidean})$ given by $x \mapsto x$ is not continuous as a function $S \to (\mathbb{R}, \text{discrete})$. Thus, passing to condensed sets has revealed a nontrivial cokernel that "explains" why $i$ is not an isomorphism.

**Remark V.5.** In the previous example, we showed that the cokernel sheaf $\underline{(\mathbb{R}, \text{Euclidean})}/\underline{(\mathbb{R}, \text{discrete})}$ is not trivial. We did so by finding $S \in \mathsf{cHaus}^{\mathrm{op}}$ on which the cokernel sheaf is nontrivial. If $S = *$, however, then $\underline{(\mathbb{R}, \text{Euclidean})}/\underline{(\mathbb{R}, \text{discrete})}(S) = 0$, and this is precisely saying that the quotient of topological abelian groups $(\mathbb{R}, \text{Euclidean})/(\mathbb{R}, \text{discrete})$ is trivial. Thus, classical theory "looks at things from the perspective of a point" while condensed theory "looks at things from the perspective of all compact Hausdorff spaces at once. (This is an informal slogan. I stole it from Logan.)

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## V.2 Presheaves

We want to fix the problem with naive condensed sets (c.f. Remark V.3). At several points, we will use *(co)limit completions of categories*. Presheaves are the terminal example. We will review the relevant theory for presheaves today, then talk about (co)limit completions in general next time.

Let $\mathsf{C}$ denote a small category. Given $X \in \mathsf{C}$, write $h_X$ for the presheaf represented by $X$, i.e. $h_X := \mathrm{Hom}_{\mathsf{C}}(-, X)$.

**Definition V.6.** Recall that a *presheaf* on $\mathsf{C}$ is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. With natural transformations as morphisms, presheaves form a category. We write this either $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ or $\mathsf{PShv}(\mathsf{C})$.

**Definition V.7.** Let $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. Its *category of elements* $\mathsf{el}\, F$ is the category defined as follows.

- The objects are morphisms of functors $h_X \to F$.[4]
- The morphisms between $(h_X \to F)$ and $(h_Y \to F)$ are the natural transformations $h_X \to h_Y$ making the following triangle commute.

$$\begin{tikzcd} h_X \arrow[rr] \arrow[dr] & & h_Y \arrow[dl] \\ & F & \end{tikzcd}$$

**Remark V.8.** The *category of elements* $\mathsf{el}\, F$ associated to the presheaf $F$ is an example of a more general construction called the *slice category*.

**Remark V.9.** The Yoneda lemma says that morphisms $X \to Y$ are in natural bijection with natural transformations $\mathrm{Hom}(-, Y) \implies \mathrm{Hom}(-, X)$. Thus, morphisms $h_X \to h_Y$ in $\mathsf{el}\, F$ are "the same thing as" certain morphisms $X \to Y$.

**Proposition V.10.** *Consider the canonical functor $D_F : \mathsf{el}\, F \to \mathsf{C}$ taking $h_X \to F$ to $X$. Let $y : \mathsf{C} \hookrightarrow \mathsf{PShv}(\mathsf{C})$ denote the Yoneda embedding. We have*

$$F \cong \varinjlim(y \circ D_F).$$

[4] I.e., the set of objects is in bijection with pairs $(X, \eta)$ where $X \in Ob(\mathsf{C})$ and $\eta$ is a natural transformation of functors $\mathrm{Hom}(-, X) \implies F$.

*Proof.* For ease of notation, write $\phi := y \circ D_F$. The functor $\phi : \mathsf{el}\, F \to \mathsf{PShv}(\mathsf{C})$ takes objects $h_X \to F$ to $h_X$ and morphisms $h_X \to h_Y$ to themselves. By construction, for every object $z \in \mathsf{el}\, F$, there is a morphism $\phi(z) \to F$ in $\mathsf{PShv}(\mathsf{C})$ and all these morphisms commute with those in the diagram $y \circ D_F$. By the universal property of colimits, there is an induced map

$$\varinjlim \phi \to F.$$

The claim is that this is an isomorphism. We will show that it is injective. That is, we will show that for every $X \in \mathsf{C}$ and $\alpha \in F(X)$, the set map $[\varinjlim \phi](X) \to F(X)$ hits $\alpha$. This boils down to the Yoneda lemma: it says that for every $\alpha \in F(X)$, there is a morphism $h_X \to F$ such that $\mathrm{id} \in h_X(X)$ maps to $\alpha$. Unwinding definitions, this implies that $\varinjlim \phi \to F$ is surjective. $\square$

# VI Lecture 6: Colimit completions

## VI.1 Presheaves as a colimit completion

As an extended motivating example, I'll explain why for a small category $\mathsf{C}$, the presheaf category $\mathsf{PShv}(\mathsf{C}) := \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ is

(1) "The same thing as" small diagrams in $\mathsf{C}$, and

(2) The "free colimit completion" of $\mathsf{C}$.

Note that these are *characterizations* of $\mathsf{PShv}(\mathsf{C})$. (Although, to make (1) a definition requires more than we talk about.) This will model our approach to other types of completions.

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First I'll explain (1). The relevant theorem is called *the density theorem*.

Let me restate (maybe a bit more clearly) what we discussed last time. Let $\mathsf{C}$ be a small category and $\mathsf{PShv}(\mathsf{C}) := \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ its presheaf category. The *Yoneda embedding* is the fully faithful functor

$$\mathcal{Y} : \mathsf{C} \hookrightarrow \mathsf{PShv}(\mathsf{C})$$

sending objects to their represented functor (i.e., $X \mapsto \mathrm{Hom}_{\mathsf{C}}(-, X)$) and morphisms $f : X \to Y$ to the natural transformation $\mathrm{Hom}_{\mathsf{C}}(-, Y) \to \mathrm{Hom}_{\mathsf{C}}(-, X)$ given by post-composition with $f$. Given a preseaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, we defined its *category of elements* $\mathsf{el}\, F$. There's a canonical functor $D_F : \mathsf{el}\, F \to \mathsf{PShv}(\mathsf{C})$.

**Proposition VI.1.** *The functor $D_F : \mathsf{el}\, F \to \mathsf{PShv}(\mathsf{C})$ factors as the composite*

$$\mathsf{el}\, F \longrightarrow \mathsf{C} \xhookrightarrow{\mathcal{Y}} \mathsf{PShv}(\mathsf{C}).$$

*Here, the first functor $\mathsf{el}\, F \to \mathsf{C}$ sends an object $h_X \to F$ to $X$ and a morphism $h_X \to h_Y$ to its corresponding morphism $X \to Y$ under the identification $\mathrm{Hom}(h_X, h_Y) \cong h_Y(X) = \mathrm{Hom}(X, Y)$.*[5] *By abuse of notation, we write $D_F$ for the functor $\mathsf{el}\, F \to \mathsf{C}$ also. You could make this more precise if you wanted to, I think.*

*Proof.* Easy exercise. (Use the Yoneda lemma.) (Is this equivalent to some version of Yoneda?) □

**Proposition VI.2** (Density theorem)**.** *A presheaf $F \in \mathsf{PShv}(\mathsf{C})$ is the colimit of its category of elements:*

$$F \cong \varinjlim_{\mathsf{el}\, F} D_F.$$

**Remark VI.3.** The density theorem and Proposition VI.1 say that every presheaf $F \in \mathsf{PShv}(\mathsf{C})$ is "canonically" presented by a diagram in $\mathsf{C}$, namely as the diagram $\mathsf{el}\, F \to \mathsf{C}$. Conversely, although a diagram $\mathsf{I} \to \mathsf{C}$ may not have a colimit in $\mathsf{C}$, we may Yoneda-embed and consider the diagram $\mathsf{I} \to \mathsf{C} \hookrightarrow \mathsf{PShv}(\mathsf{C})$. *Since* $\mathsf{PShv}(\mathsf{C})$ *is complete and cocomplete, this diagram admits a colimit* (which is a presheaf on $\mathsf{C}$). In some weak sense, we're saying that presheaves on $\mathsf{C}$ are "the same thing as" diagrams in $\mathsf{C}$.

Did Michael talk about this?

Let me just tack on the (co)completeness property I just mentioned.

**Proposition VI.4** (Tag 00VB)**.** *If $\mathsf{C}$ is any small category, then $\mathsf{PShv}(\mathsf{C})$ is complete and cocomplete. Moreover, limits and colimits of presheaves are "computed pointwise" in the sense that given a diagram $\mathcal{F} : \mathsf{I} \to \mathsf{PShv}(\mathsf{C})$,*

- *The presheaf $c \mapsto \varinjlim_i \mathcal{F}_i(c)$ is the colimit of $\mathcal{F}$, and*
- *The presheaf $c \mapsto \varprojlim_i \mathcal{F}_i(c)$ is the limit of $\mathcal{F}$.*

*We may summarize this by writing $[\varprojlim_i \mathcal{F}_i](c) = \varprojlim_i [\mathcal{F}_i(c)]$ and $[\varinjlim_i \mathcal{F}_i](c) = \varinjlim_i [\mathcal{F}_i(c)]$.*

---

Here's (2).

[5] That identification is the Yoneda lemma.

**Proposition VI.5** (PShv(C) is the free colimit completion)**.** *By Proposition VI.2, each presheaf $F \in \mathsf{PShv}(\mathsf{C})$ is canonically a colimit over the diagram of its category of elements: $F \cong \varinjlim[y \circ D_F]$. Suppose as given a cocomplete category* D*. For each functor $G : \mathsf{C} \to \mathsf{D}$, define a functor $\overline{G}$: $\mathsf{PShv}(\mathsf{C}) \to \mathsf{D}$ by*

$$\left(\varinjlim[y \circ D_F]\right) \mapsto \varinjlim F D_F.$$

*Claim: the functor $\overline{G}$ is cocontinuous and the functor $G \mapsto \overline{G}$ defines a categorical equivalence*

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}) \cong \mathsf{Fun}^{cocts}(\mathsf{PShv}(\mathsf{C}), \mathsf{D}).$$

**Remark VI.6.** We already knew that PShv(C) had all colimits (and limits). This proposition says that PShv(C) is *universal* with respect to this property: any functor $\mathsf{C} \to \mathsf{D}$ landing in a cocomplete category factors uniquely through a cocontinuous functor $\mathsf{PShv}(\mathsf{C}) \to \mathsf{D}$.

## VI.2 So, what's a colimit completion?

Suppose as given a small category C. We gave three interpretations of presheaves on C:

(1) Presheaves are functors $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ (this was our definition);

(2) Presheaves "are" diagrams in C (we only formalized one direction of this); and

(3) Presheaves are the free colimit completion of C (this is a universal property, thus a characterization).

Now we ask, what if we only want to complete C at "nice" colimits? The above suggests three approaches we may try, which are all useful. Namely, a completion of C at "nice" colimits should be...

(1) The subcategory of "nice" presheaves $\mathsf{S} \subsetneq \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$;

(2) The subcategory of "nice" diagrams in C; and

(3) The free "nice colimit" completion of C.

We are interested in the *filtered colimit* completion, its dual *cofiltered limit* completion, and the *sifted colimit* completion. We will define/construct these along the lines of (1), (2), and/or (3) above.

## VI.3 Ind(C)–completing at filtered colimits

Here we develop the our first "new" type of completion. We make our definition in the likeness of (2), taking "nice"= *small, filtered diagrams*. We will write Ind(C) for the resulting category. This will be a *free completion at filtered colimits*, giving us our characterization (3). As for (1), we will exhibit a canonical embedding $\mathsf{Ind}(\mathsf{C}) \hookrightarrow \mathsf{PShv}(\mathsf{C})$ and prove that it is fully-faithful, with essential image = the full subcategory spanned by filtered colimits of representables. (If C admits finite colimits, then we can say more: filtered colimits are precisely those commuting with finite limits, and we will deduce that Ind(C) is precisely the subcategory of *right exact* presheaves $\mathsf{Fun}^{rex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$. I.e., those which preserve finite colimits.)

**Definition VI.7.** A diagram I is called *filtered* if (i) it is nonempty, (ii) for any two objects $j, j'$ there exist arrows $j \to k \leftarrow j'$, and (iii) for any parallel arrows $a, b : j \to j'$ there exists an arrow $j' \to k$ equalizing them. A colimit is called *filtered* if it is taken over a filtered diagram.

**Remark VI.8.** In Set, finite limits commute with filtered colimits. In fact, this characterizes filtered diagrams: a category I is filtered if and only if for every diagram $F : \mathsf{I} \to \mathsf{Set}$, its colimit commutes with all finite limits: that means for every finite diagram $G : \mathsf{J} \to \mathsf{C}$, one has

$$\varprojlim_{\mathsf{J}} \varinjlim_{\mathsf{I}} F \times G \cong \varinjlim_{\mathsf{I}} \varprojlim_{\mathsf{J}} F \times G.$$

**Definition VI.9.** An *ind-object* of C is a filtered diagram $\mathsf{I} \to \mathsf{C}$. Ind-objects are sometimes written "$\varinjlim_{\mathsf{I}}$". Lurie also just writes $\{x_i\}_{\mathsf{I}}$.

**Remark VI.10.** What should morphisms be? We want to think of ind-objects as "stand-ins" for their colimits, i.e. as "formal filtered colimits." (An ind-object may not really have a colimit in $\mathsf{C}$, but $\mathcal{Y}$ embeds them fully faithfully into $\mathsf{PShv}(\mathsf{C})$, which has all colimits.) Then morphisms should be such that these "formal filtered colimits" behave like actual colimits. To be precise: let us regard two ind-objects $X :=$ "$\varinjlim_{\mathsf{I}}$"$x_i$, $Y :=$ "$\varinjlim_{\mathsf{J}}$"$y_j$ as "formal colimits" in $\mathsf{C}$. We ought to have

$$\mathrm{Hom}(X, Y) = \varprojlim_{\mathsf{I}} \mathrm{Hom}(x_i, Y) = \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathrm{Hom}(x_i, y_j).$$

We expect the first equality because $\mathrm{Hom}$ takes colimits to limits in the first variable. We expect the second equality for some slightly more complicated reason. This brings us to our definition.

**Definition VI.11.** The *ind-morphisms* between two ind-objects $\{x_i\}_{\mathsf{I}} \to \{y_j\}_{\mathsf{J}}$ are defined as

$$\mathrm{Hom}(\{x_i\}_{\mathsf{I}}, \{y_j\}_{\mathsf{J}}) := \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathrm{Hom}(x_i, y_j).$$

**Definition VI.12** (Characteriztion (2), $\mathsf{Ind}(\mathsf{C})$ as diagrams)**.** We denote by $\mathsf{Ind}(\mathsf{C})$ the *ind-category* of $\mathsf{C}$.

**Proposition VI.13** (Characterization (1), $\mathsf{Ind}(\mathsf{C})$ as presheaves)**.** *Consider the canonical functor $F$ : $\mathsf{Ind}(\mathsf{C}) \to \mathsf{PShv}(\mathsf{C})$ that Yoneda embeds an ind-object $X := \{x_i\}_{\mathsf{I}}$ and takes its colimit in $\mathsf{PShv}(\mathsf{C})$. This functor is fully faithful. Thus, $\mathsf{Ind}(\mathsf{C})$ may be identified with the full subcategory of $\mathsf{PShv}(\mathsf{C})$ spanned by ind-objects, i.e. spanned by the filtered colimits of representable presheaves.*

*Proof.* This is basically by definition. Let $X, Y$ be ind-objects. Recall that $\mathrm{Hom}$ takes colimits to limits in the first argument and that limits of presheaves are computed pointwise. We get a natural isomorphism

$$\mathrm{Hom}_{\mathsf{PShv}(\mathsf{C})}(\underbrace{\varinjlim_{\mathsf{I}}[-, x_i]}_{FX}, \underbrace{\varinjlim_{\mathsf{J}}[-, y_j]}_{FY}) \cong \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathrm{Hom}(x_i, y_j).$$

The right-hand side is $\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(X, Y)$ by definition. Then (by definition) the functor $\mathsf{Ind}(\mathsf{C}) \to \mathsf{PShv}(\mathsf{C})$ is just the identity on hom-sets, which is bijective. $\square$

**Remark VI.14.** Conside the embedding $\mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C})$ which maps an object $c$ to its singleton diagram $* \mapsto c$. This clearly factors the Yoneda embedding, i.e. $\mathcal{Y}$ lands in $\mathsf{Ind}(\mathsf{C})$, i.e. the following diagram commutes.

$$\begin{tikzcd}
 & \mathsf{Ind}(\mathsf{C}) \arrow[d, hook] \\
\mathsf{C} \arrow[ur, hook] \arrow[r, hook, "\mathcal{Y}"'] & \mathsf{PShv}(\mathsf{C})
\end{tikzcd}$$

---

We said that $\mathsf{Ind}(\mathsf{C})$ is naturally the full subcategory of $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ of filtered colimits of representables. If $\mathsf{C}$ is "nice," we can describe this subcategory more concretely.

**Corollary VI.15** (What kind of presheaves?)**.** *Recall that filtered colimits are precisely those commuting with finite limits. Suppose that $\mathsf{C}$ is finitely cocomplete, so that $\mathsf{C}^{\mathrm{op}}$ admits finite limits. [Don't we want $\mathsf{C}^{\mathrm{op}}$ to have finite colimits?] The previous corollary says that an ind-object $X$ is a filtered colimit of representables $X = \varinjlim_{\mathsf{I}} \mathcal{Y}x_i$. Then we may evaluate $X$ at a finite colimit in $\mathsf{Set}$ and compute:*

$$\begin{aligned}
X(\varinjlim_{\mathsf{J}} y_j) &= [\varinjlim_{\mathsf{I}} \mathrm{Hom}(-, x_i)](\varinjlim_{\mathsf{J}} y_j) \\
&= \varinjlim_{\mathsf{I}} \mathrm{Hom}(\varinjlim_{\mathsf{J}} y_j, x_i) \\
&= \varinjlim_{\mathsf{I}} \varprojlim_{\mathsf{J}} \mathrm{Hom}(y_j, x_i) \\
&= \varprojlim_{\mathsf{J}} \varinjlim_{\mathsf{I}} \mathrm{Hom}(y_j, x_i).
\end{aligned}$$

*In order, these equalities follow from: (1) definition; (2) colimits of presheaves are computed pointwise; (3)* Hom *takes colimits to limits in the first argument; and (4) filtered colimits commute with finite limits. Thus, ind-objects are presheaves which preserve finite colimits.*

**Proposition VI.16.** *The converse is also true: if a presheaf $F$ preserves finite colimits, then $F$ is a filtered colimit of representables. (Namely, $D_F$ is filtered, whence $F \cong \varinjlim D_F$ proves the claim.)*

**Corollary VI.17.** *Thus,* $\mathsf{Ind}(\mathsf{C})$ *is precisely the full subcategory of right exact presheaves, i.e. those preserving finite colimits. Then we may identify (or take as defintion)*

$$\mathsf{Ind}(\mathsf{C}) \cong \mathsf{Fun}^{rex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set}).$$

---

Lastly, we want a characterization of $\mathsf{Ind}(\mathsf{C})$ in the likeness of (3), a "free filtered colimit completion" universal property.

**Proposition VI.18** (Characterization (3), $\mathsf{Ind}(\mathsf{C})$ as free filtered colimit completion)**.** *If* $\mathsf{D}$ *has filtered colimits, then restriction along* $\mathcal{Y} : \mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C})$ *induces an equivalence of categories*

$$\Theta : \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \cong \mathsf{Fun}^{filco}(\mathsf{Ind}(\mathsf{C}), \mathsf{D}).$$

*Proof.* Restriction along Yoneda is the functor $\Theta : \mathsf{Fun}^{filco}(\mathsf{Ind}(\mathsf{C}), \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{D})$ given by $F \mapsto \mathcal{Y}F$. We will show that $\Theta$ is fully faithful and essentially surjective. Suppose as given $F, G \in \mathsf{Fun}^{filco}(\mathsf{Ind}(\mathsf{C}), \mathsf{D})$.

(Full) Take as given a natural transformation $\eta : \mathcal{Y}F \to \mathcal{Y}G$. We want to extend this to a natural transformation $\bar{\eta} : F \to G$. Recall that $X \in \mathsf{Ind}(\mathsf{C})$ has a canonical presentation as a filtered colimit $X = \varinjlim_{\mathsf{I}} \mathcal{Y}x_i$. Since $F$ preserves filtered colimits, $FX = \varinjlim_{\mathsf{I}} F\mathcal{Y}x_i$. Likewise for $GX$. Now the existing $\eta_{x_i} : F\mathcal{Y}x_i \to G\mathcal{Y}x_i$ determine a map $FX \to FY$. All these assemble to a natural transformation $\bar{\eta}$ which obviously satisfies $\Theta(\bar{\eta}) = \eta$, i.e. $\bar{\eta}$ restricts to $\eta$.

(Faithful)

(Essentially surjective) $\square$

**Remark VI.19.** In other words, $\mathsf{Ind}(\mathsf{C})$ has the following property: for any functor $F : \mathsf{C} \to \mathsf{D}$ to a category admitting filtered colimits, there exists a unique functor $\mathsf{Ind}(\mathsf{C}) \to \mathsf{D}$ such that (1) it extends $F$ along $\mathcal{Y} : \mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C})$, and (2) it preserves filtered colimits. And this describes a bijection (in fact, a functorial equivalence) between $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ and $\mathsf{Fun}^{filco}(\mathsf{Ind}(\mathsf{C}), \mathsf{D})$.

# VII Lecture 7: Profinite sets and spaces

Let $\mathsf{C}$ be a category. Last time we defined its *ind-completion* $\mathsf{Ind}(\mathsf{C})$. I wanted to define other completions, but we did not have time, but that is OK since we described the general procedure for completions. Today, we will use a completion "dual" to $\mathsf{Ind}(\mathsf{C})$, and will finally do something vaguely condensed.

## VII.1 $\mathsf{Pro}(\mathsf{C})$–completing at cofiltered limits

**Definition VII.1.** Given a category $\mathsf{C}$, we define its *pro-category* $\mathsf{Pro}(\mathsf{C})$ as

$$\mathsf{Pro}(\mathsf{C}) := (\mathsf{Ind}\,\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}.$$

**Proposition VII.2.** *The pro-category* $\mathsf{Pro}(\mathsf{C})$ *is "dual" to* $\mathsf{Ind}(\mathsf{C})$ *in many ways:*

*(1)* $\mathsf{Pro}(\mathsf{C})$ *is the category of "formal cofiltered limits" in* $\mathsf{C}$*, i.e. cofiltered diagrams* $\mathsf{I} \to \mathsf{C} \hookrightarrow \mathsf{PShv}(\mathsf{C})$ *with appropriate morphisms. Namely, hom-sets have a description dual to Definition VI.11:*

$$\mathrm{Hom}_{\mathsf{Pro}(\mathsf{C})}(\text{“}\varprojlim\text{”}_{\mathsf{I}} x_i, \text{“}\varprojlim\text{”}_{\mathsf{J}} y_j) = \varprojlim_{\mathsf{J}} \operatorname*{colim}_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(x_i, y_j).$$

*(2) We may identify* $\mathsf{Pro}(\mathsf{C})$ *with its image under the embedding* $\mathsf{Pro}(\mathsf{C}) \hookrightarrow \mathsf{PShv}(\mathsf{C})$ *given by Yoneda-embedding a diagram and taking its limit in* $\mathsf{PShv}(\mathsf{C})$*. This identifies* $\mathsf{Pro}(\mathsf{C})$ *as the full subcategory spanned by pro-objects, i.e. cofiltered limits of representable presheaves.*

*(3) In particular, dual to Corollary VI.17, if* $\mathsf{C}$ *is finitely complete, then* $\mathsf{Pro}(\mathsf{C})$ *is naturally the full subcategory of left exact presheaves, i.e. those preserving finite limits:*

$$\mathsf{Pro}(\mathsf{C}) \cong \mathsf{Fun}^{lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set}).$$

*(4)* $\mathsf{Pro}(\mathsf{C})$ *has the "free completion of* $\mathsf{C}$ *at cofiltered limits" universal property.*

## VII.2 Basic structure of $\mathsf{ProFinSet}$

Now take $\mathsf{C} = \mathsf{FinSet}$ and $\mathsf{D} = \mathsf{Top}$. Let *disc*: $\mathsf{FinSet} \to \mathsf{Top}$ endow each finite set with the discrete topology. Since $\mathsf{Top}$ is complete, the universal property says that $disc$ has a unique cofiltered-limit preserving extension

$$\overline{disc} : \mathsf{ProFinSet} \to \mathsf{Top}.$$

In our formulation of $\mathsf{PShv}(\mathsf{C})$'s free colimit completion property [c.f. Proposition VI.5] we explicitly constructed (from a functor to a cocomplete category $\mathsf{C} \to \mathsf{D}$) the colimit-preserving extension $\mathsf{PShv}(\mathsf{C}) \to \mathsf{D}$. To prove that $\mathsf{Ind}(\mathsf{C})$ and $\mathsf{Pro}(\mathsf{C})$ have their universal properties, one may carry out a totally analogous construction (to construct an inverse to the $\mathcal{Y}$-restriction functor). In the present case, we get a concrete definition of $\overline{disc}$: it acts by $\{s_i\}_{\mathsf{I}} \mapsto \{disc(s_i)\}_{\mathsf{I}} \mapsto \varprojlim_{\mathsf{I}} disc(s_i)$. All this summarizes to the following commutative diagram.

$$\begin{tikzcd}
 & \mathsf{ProFinSet} \arrow[d, dashed, "\overline{disc}"] \arrow[dr, "\{s_i\}_i \mapsto \{disc(s_i)\}_i"] & \\
\mathsf{FinSet} \arrow[ur] \arrow[r, hook, "disc"'] & \mathsf{Top} & \mathsf{Pro}(\mathsf{Top}) \arrow[l, "\{X_j\}_j \mapsto \varprojlim X_j"]
\end{tikzcd}$$

We are gathered here today to present two basic and nice structure theorems.

**Theorem VII.3.** *The functor* $\overline{disc} : \mathsf{ProFinSet} \to \mathsf{Top}$ *is continuous and fully faithful.*

**Theorem VII.4.** *Call a space* *profinite* *if it belongs to the essential image of* $\overline{disc}$*. TFAE.*

*(1)* $X$ *is a profinite space.*

(2) $X$ *is a cofiltered limit of finite, discrete spaces.*

(3) $X$ *is a limit of finite, discrete spaces.*

(4) $X$ *is a Stone space, i.e. it is Hausdorff, compact, and totally disconnected.*

**Remark VII.5.** Let's ruminate on Theorem VII.3. Suppose that $S$ is a profinite set. The space $X := \overline{disc}(S)$ does not "know" it arises from a profinite set—a priori, to remember $X \cong \overline{disc}(S)$ is "extra structure," comparable to a filtration. Nevertheless, what Theorem VII.3 tells us is that there is really no loss of information. Up to isomorphism, the topology on $X$ uniquely encodes the profinite set $S$. Say this better?

## VII.3 The profinite topology

Let's lay a bit of groundwork and prove part of Theorem VII.4. If I say something wrong or confusing, maybe consult Section 08ZS.

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The functor $\overline{disc}$ maps a profinite set $\{s_i\}$ to the colimit of each set $s_i$ regarded as a discrete space. What is this colimit (both its underlying set and topology)? We talked about this a bit in Lecture II?

**Proposition VII.6.** *The forgetful functor* $U : \mathsf{Top} \to \mathsf{Set}$ *commutes with all small limits and colimits.*

*Proof.* There are a few ways you could do this. Maybe the shortest: the forgetful functor is left and right adjoint (to the indiscrete space functor and to $disc$, respectively). Left adjoints always commute with colimits. Right adjoints always commute with limits. ☐

**Proposition VII.7.** *Let* $\{X_i\}_\mathsf{I}$ *be a pro-system of topological spaces. By Proposition VII.6, the underlying set of* $\varprojlim X_i$ *is given by*

$$U \varprojlim X_i = \{(x_i)_i \in \prod_i X_i : \forall \phi : i \to i', \text{ we have } (X\phi)(x_i) = x_{i'}\}$$

*where the right-hand side is the general formula for a limit of sets as in Example II.19. Define the profinite topology on* $U \varprojlim X_i$ *to be the subspace topology induced by* $U \varprojlim X_i \hookrightarrow \prod X_i$*. The claim is that this is the smallest*[6] *topology on* $U \varprojlim X_i$ *such that every projection* $p_j : \varprojlim X_j \to X_j$ *is continuous. In effect,* $(U \varprojlim X_i, \text{profinite topology})$ *is the limit of the pro-system* $\{X_i\}$*.*

Summarizing, Proposition VII.6 describes the *underlying set* of a limit or colimit of spaces in general, and Proposition VII.7 concretely describe the topology on a *cofiltered limit* of spaces. In particular, given $S = \{s_i\} \in \mathsf{ProFinSet}$, we can concretely describe $\overline{disc}(S)$, since $\overline{disc}$ just takes the cofiltered limit of the $s_i$ as discrete spaces. This concrete description lets us use our hands to prove the following.

**Proposition VII.8.** *If* $\{X_i\}_i$ *is a pro-system, then* $X := \varprojlim X_i$ *with the profinite topology is compact, Hausdorff, and totally disconnected.*

*Proof.* That $X$ is Hausdorff is clear: each $X_i$ is Hausdorff, so then $\prod X_i$ is, so then are its subspaces, in particular $X$ is Hausdorff. (This holds for general limits of Hausdorff spaces.)

Proof that $X$ is compact: Note that $X$ is a subspace of a product of compact spaces (the discrete spaces $X_i$). Tychonoff's theorem says that the product is compact, so it suffices to show that $X$ is closed. (To-do: show that) (This holds for general limits of compact Hausdorff spaces.)

Proof that $X$ is totally disconnected: let $a, b \in X$ be distinct, which means there is some $p_j : X \to X_j$ such that $p_j(a) \neq p_j(b)$. Choose sets $A, B$ such that $A, B$ partition $X$, $a \in A$, and $b \in B$. Since $p_j$ is surjective, $\pi_j^{-1}(A)$ and $p_j^{-1}(B)$ partition $X$, and they are open because $A, B \subseteq X_j$ are open, and seperate $a$ and $b$ by definition. Thus there are no connected subsets with more than one element. (This holds for general limits of totally disconnected spaces.) ☐

**Corollary VII.9.** *We conclude that (1)* $\implies$ *(4) in Theorem VII.4. For by Proposition VII.7, if* $\{X_i\}$ *is a pro-system of finite sets, then* $X := \varprojlim X_i$ *with the profinite topology is a limit in* $\mathsf{Top}$ *of* $\{X_i\}$ *where the* $X_i$ *carry the discrete topology. Therefore* $F(\{X_i\}) \cong (X, \text{profinite topology})$*. The latter is Hausdorff, compact, and totally disconnected by Proposition VII.8.*

[6] Least open sets.

## VII.4 $\overline{disc}$ is continuous and fully faithful

**Proposition VII.10.** ProFinSet *has all limits.*

*Proof.* A category has finite limits $\iff$ it has equalizers and finite products. Furthermore, a category with finite limits and cofiltered limits has all limits. We know that FinSet has equalizers and finite products, hence finite limits. This generally passes over to the procategory, so ProFinSet has finite limits. It also has cofiltered limits by construction, hence it is complete. □

**Proposition VII.11.** *The functor* $\overline{disc}$ : ProFinSet $\to$ Top *induced by* $disc$ : FinSet $\hookrightarrow$ Top *preserves limits.*

*Proof.* I'll give two (sketch) proofs.

First proof: we can directly verify that $\overline{disc}$ preserves equalizers and finite products, so $\overline{disc}$ preserves finite limits. (See Michael's paper.) Additionally, the dual to Proposition VI.18 says that $\overline{disc}$ preserves cf limits. Since $\overline{disc}$ preserves finite and cf limits, it preserves all limits.

Second proof: recall that $\overline{disc}$ is the composite ProFinSet $\to$ Pro(Top) $\to$ Top. The first map is induced by $disc$ and the second is "take the limit." We'll show that both maps are continuous:

- Since FinSet $\hookrightarrow$ Top preserves finite limits, so does ProFinSet $\to$ Pro(Top). Furthermore, ProFinSet $\to$ Pro(Top) preserves cofiltered limits,[7] and we conclude ProFinSet $\to$ Pro(Top) preserves all limits.
- Since Top has cf limits (it is complete), the dual to **??** says that evaluation Pro(Top) $\to$ Top is a left adjoint, hence preserves limits.

□

Right adjoints have the great property that they preserve all limits. The last proposition says that $\overline{disc}$ does too. The adjoint functor theorem gives us the converse: $\overline{disc}$ has a left adjoint.

**Corollary VII.12** (Application of adjoint functor theorem)**.** *Denote by* $\hat{\pi}_0$: Top $\to$ ProFinSet *the functor given by* $X \mapsto \{X_i\}$ *where* $\{X_i\}$ *is the codirected set of finite, discrete quotients of* $X$. *It is left adjoint to* $\overline{disc}$.

So, we have an adjunction

$$\text{ProFinSet} \underset{\hat{\pi}_0}{\overset{\overline{disc}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \text{Top}.$$

The counit of this adjunction is a natural transformation $\epsilon : \hat{\pi}_0 \varprojlim(-) \implies \mathrm{id}_{\mathsf{ProFinSet}}$. In general, a left adjoint is fully faithful $\iff$ its counit is an isomorphism. And we can show our counit $\epsilon$ is an isomorphism.

**Proposition VII.13.** *The counit of the adjunction* $F \dashv \hat{\pi}_0$ *is an isomorphism. Equivalently,* $\overline{disc}$ *is fully faithful.*

*Proof.* I said this was elementary but it's not very short. And I'm not convinced it's very useful. Consult Chicago course notes. It's an application of elementary facts about compact and finite spaces. □

## VII.5 Profinite sets $\cong$ Stone spaces

We've shown that $\overline{disc}$ : ProFinSet $\to$ Top is fully faithful and lands in Stone spaces (compact, Hausdorff, totally disconnected). Now we will show that all Stone spaces arise as profinite spaces. Thus $\overline{disc}$ is an equivalence between ProFinSet and Stone spaces.

**Proposition VII.14.** *If* $X$ *is a compact, Hausdorff, totally disconnected space, then* $X$ *is a profinite space, i.e. it belongs to the essential image of* $\overline{disc}$.

**Corollary VII.15.** *We have that (4)* $\implies$ *(1) in Theorem VII.4.*

[7] Given any functor $G$, the map on procategories induced by $G$ preserves cofiltered limits; this is [KS, Prop 6.1.9].

*Proof.* Take $X$ as given. Denote by $\mathrm{FinPar}(X)$ the set of finite partitions of $X$, i.e. each element $A \in \mathrm{FinPar}(X)$ consists of finitely many disjoint, nonempty opens $U_j \subseteq X$ such that $X = \bigsqcup_{U_j \in A} U_j$. If $A, B \in \mathrm{FinPar}(X)$, declare that $A \leq B$ iff $B$ refines $A$. This makes $\mathrm{FinPar}(X)$ into a poset.

Consider the diagram $\mathrm{FinPar}(X) \to \mathsf{Top}$ giving each $A \in \mathrm{FinPar}(X)$ the discrete topology. Define

$$X' := \varprojlim_{A \in \mathrm{FinPar}(X)} A$$

To be the limit over this diagram. Note that $X'$ is profinite since $\mathrm{FinPar}(X)$ is cofiltered. For each $A \in \mathrm{FinPar}(X)$, denote by $f_A : X \to A$ the map which sends $x \in X$ to the open $U \in A$ containing it. The collection of $f_A$'s induces a map $f : X \to X'$.

Since $X'$ is profinite, it is compact. So, if we show that $f : X \to X'$ is a bijection, we can conclude it is a homeomorphism.

Injectivity of $\overline{disc}$ is easy: since $X$ is totally disconnected, if $x, y \in X$, any cover $A \in \mathrm{FinPar}(X)$ either separates $x$ and $y$ or can be refined to a cover separating $x$ and $y$. This suffices to find that $\overline{disc}$ is injective.

(To-do: show surjective.) $\square$

# VIII Lecture 8: The site $\mathsf{cHaus}_\kappa$

We're going to start building to some proper condensed math now.

**Definition VIII.1.** A strong limit cardinal is a cardinal $\kappa$ so that $\lambda < \kappa$ implies $2^\lambda < \kappa$.

For every uncountable strong limit cardinal $\kappa$, we define $\mathsf{cHaus}_\kappa$ to be the category of compact Hausdorff spaces of cardinality strictly less than $\kappa$. Note that $\mathsf{cHaus}_\kappa$ is then essentially small.

One can prove that $\mathsf{cHaus}_\kappa$ has all finite limits: just take the finite limit in $\mathsf{cHaus}$, and you won't exceed the cardinality bound.

$\mathsf{cHaus}_\kappa$ is finitely cocomplete. In fact, $\mathsf{cHaus}_\kappa$ is closed under all colimits whose diagrams have fewer than $\lambda$ objects, where $\lambda$ is the cofinality of $\kappa$. Here, the cofinality is the smallest $\lambda$ so that there exist a collection of $\lambda$ many sets of $\kappa$-small cardinality but whose disjoint union has cardinality $\kappa$.

*Proof.* The inclusion $i : \mathsf{cHaus} \to \mathsf{Top}$ is right adjoint to Stone-Cech compactification; left adjoints preserve colimits, and so we can get a colimit in $\mathsf{cHaus}$ by taking the ordinary colimit and then applying the Stone-Cech compatification. Thus $\mathsf{cHaus}$ is cocomplete, and so $\mathsf{cHaus}_\kappa$ is finitely cocomplete since

$$|\beta X| \leq 2^{2^{|X|}} < \kappa$$

whenever $|X| < \kappa$.

In fact, $\mathsf{cHaus}_\kappa$ admits all $\lambda$-small coproducts and all coequalizers, so we get the $\lambda$ part. □

We give $\mathsf{cHaus}_\kappa$ a Grothendieck topology by families of finite jointly surjective morphisms.

## VIII.1 Weakly contractible objects

Let $\mathcal{C}$ be a site. An object $X \in \mathcal{C}$ is said to be *weakly contractible* if, for every epimorphism $\mathcal{F} \to \mathcal{G}$ of sheaves of sets, the map $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective.

We say that $\mathcal{C}$ has enough weak contractibles if every object can be covered by weakly contractible objects.

If $X$ is weakly contractible, then $\Gamma_X : \mathsf{Ab}(\mathcal{C}) \to \mathsf{Ab}$ is exact, and $H^p(X, \mathcal{F}) = 0$ for all $p \geq 1$ and all abelian sheaves.

*Proof.* The second part is implied immediately by the first; the first is definitional. □

**Corollary VIII.2.** *If $\mathcal{C}$ is a cite with enough weakly contractible objects, then $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact if and only if $\mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X)$ is exact for every weakly contractible $X$. Furthermore, products are exact in $\mathsf{Ab}(\mathcal{C})$, and in any inverse system*

$$\cdots \to \mathcal{F}_3 \to \cdots \to \mathcal{F}_1,$$

*the projection map*

$$\varprojlim_n \mathcal{F}_n \to \mathcal{F}_1$$

*out of the inverse limit is surjective.*

*Proof.* Follow your nose. todo: say more. □

## VIII.2 Extremally disconnected spaces