2024 Notebook

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I January

> Prior to the advent of the brain, there was no color and no sound in the universe, nor was there any flavor or aroma...before brains the universe was also free of pain and anxiety.
>
> *Roger Sperry*

New year. And a new notebook. Last one was getting tedious to compile, and it was about time I tweak the tex anyway. I should start hearing back from graduate schools soon. I am pretty excited to start grad school, and now that the anxiety of applications is gone, I can feel and and appreciate how much time six years (the duration I plan to be in graduate school) is to do things. This notebook remains a documentation of what I'm studying in homotopy theory.

I.1 (1/1) Cardinality

A "set of all sets" is set-theoretically impossible, although there is a proper class T of such things. A group is a set, hence *the class of groups of order* 2 is at least the size of T since each set $S \in T$ gives rise to a group of order 2, namely $\{S, S\}$ with whatever[1] group structure you'd like. In particular, there is a *proper* class of groups of order 2. The lesson: for many purposes, we should count up to isomorphism.

Let X be a finite set and $\sim$ an equivalence relation. The quotient $X/\sim$ is finite, hence its *cardinality* is well-defined and equals the number of equivalence classes, which we can count "over X" via the easy formula

$$|X/\sim| = \sum_{x\in X} \frac{1}{|[x]|}.$$

Notice this formula does not use the relation $\sim$ in any meaningful way (it is just a weird way of writing $|X/\sim| = \sum_{[x]} 1$). In certain situations, we may not like this, and let me explain a better and similar-looking quantity to call "the size of $X/\sim$."

Observation I.1. Let X be a finite set and $\sim$ an equivalence relation. *A priori*, the natural distribution on a finite set X is the uniform one. You can also put the uniform distribution on $X/\sim$, but this does not agree with another distribution we can put on $|X/\sim|$, namely that where $\mathbb{E}([x])$ is induced by the likelihood of obtaining any $y \in [x]$.

Example I.1. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on the set

$$X := \{1, 2, 3, 4, 5\}$$

as the cycle (15)(24). Then X/G has three points, those being the orbits $\{1, 5\}$, $\{2, 4\}$, and $\{3\}$. If we take the uniform distribution on X, then the induced distribution on X/G is *not* uniform. Rather, it weighs the listed orbits $2/5$, $2/5$, and $1/5$, respectively, but the uniform distribution would weigh them each $1/3$.

Remark I.1. Our use of probability here is not essential. It just gives a natural language to talk about the "size" of objects—bigger objects should take up more of the distribution.

Remark I.2. Let G be any finite group acting on a finite set X. If the action is free, then there is no issue like the above, for freeness is equivalent to trivial stabilizers, which implies that the orbits of G have uniform size, namely the orbits are of size $|G|$.

Remark I.3. The category of finite sets is the "categorification" of the natural numbers. We can divide two natural numbers—how to divide two finite sets S and T? If T is a finite group acting on S, we may consider the quotient S/T. By the previous remark, if this action is free, then $|S/T| = |S|/|T|$. Hence, quotienting sets by finite *free* actions is a "categorification" of division. However, this process only produces finite sets, which is not what we want because finite sets are categorified natural numbers—we should obtain *rationals* via division! The solution is to consider all action quotients (possibly non-free!). But this suffers from the same problem, namely that a non-free quotient is still *just* a finite set (e.g., $|X/G| = 3$ above). Hm...

[1] Which ever. Of the two.

The problem is that $3 \in X$ has more automorphisms (a smaller orbit), so it "appears larger" in X/G than it "really is" because the set X/G does not see automorphisms. This suggests that the "size" of an element should be inversely proportional to its number of automorphisms. We therefore should replace sets with objects that carry automorphism data (groupoids) and extend the notion of cardinality to account for that data.

Definition I.1. Let X be an ordinary groupoid. Define its *homotopy cardinality* as the sum

$$|X| = \sum_{[x]\in\pi_0 X} \frac{1}{|\mathrm{Aut}(x)|}.$$

Example I.2. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on a five-element X as the cycle $\sigma = (15)(24)$ again. Consider the associated *action groupoid* $X//G$ [nLab]. It has $\mathrm{Ob}(X//G) = X$ and a morphism $g : x \to y$ if and only if $\exists g : gx = y$. In this case, we can compute its homotopy cardinality as

$$|X| = \frac{1}{|\mathrm{Aut}(1)|} + \frac{1}{|\mathrm{Aut}(2)|} + \frac{1}{|\mathrm{Aut}(3)|} = \frac{1}{|\{\sigma^2 = \mathrm{id}\}|} + \frac{1}{|\{\sigma^2 = \mathrm{id}\}|} + \frac{1}{|\{\mathrm{id},\sigma\}|} = 5/2.$$

This works. In the above example, it weights elements in the quotient correctly, namely prescribing 3 a weight of $1/2$. More generally, given any (possibly non-free) finite group action $G \curvearrowright X$, we have $|X//G| = |X|/|G|$ as desired. You may crank out exotic examples. You may ponder the cardinality of the core of your favorite category. For example, $|\mathsf{FinSet}^{\cong}| = e$.

Homotopy cardinality is homotopy invariant. Furthermore, it has the essential properties of ordinary cardinality: it is additive and multiplicative over products and coproducts, respectively. How to extend this notion to ∞-groupoids? Let me just give you the definition: for an ∞-groupoid X, we define its *homotopy cardinality* as the sum

$$|X| := \sum_{[x]\in\pi_0 X} \prod_{n=1}^{\infty} |\pi_n(X,x)|^{(-1)^k} = \sum_{[x]\in\pi_0 X} \frac{|\pi_2(X,x)| \cdot |\pi_4(X,x)| \cdots}{|\pi_1(X,x)| \cdot |\pi_3(X,x)| \cdots}.$$

This satisfies a more general multiplicativity property: given a fibration $F \to E \to B$ over a connected base B, the homotopy long exact sequence yields $|E| = |F||B|$. (This is more general because fiber bundles are "twisted" cartesian products.) This tells us, for instance, that

$$|BG| = \frac{1}{|G|}.$$

Remark I.4. What's with the alternation in the product? Crudely, it is a manifestation of an iterated inclusion-exclusion argument. I think of this informally: the cardinality of the underlying set of X is the number of its points. But homotopy theory says to replace sets with 0-groupoids, in which case the cardinality is the number o f connected components. For simplicity, suppose X is connected; then its underlying 1-groupoid has one point up to isomorphism, but that point may have automorphisms, accounted for by $\pi_1 X$, and we know that cardinality should be inversely proportional to this. But wait—those automor mtomorphisms "smaller." But wait—the automorphisms of automorphisms have automorphisms, accounted for by $\pi_3 X$, which by parallel reasoning make $|X|$ *smaller*. So on and so forth *ad infinitum*.

Remark I.5. We are implicitly assuming the defining series for $|X|$ converges. Call spaces for which this is the case *tame*.

Recall that a space X is called *n-finite* if $\pi_{k>n} X = 0$ and $\pi_{k\le n} X$ is finite, and that X is called *π-finite* if it is n-finite for some n. It is clear that π-finite spaces are tame. Thus, π-finite spaces seem like a good category of spaces in which to think about homotopy cardinality.

Definition I.2. Write $\mathsf{S}_{\mathrm{fin}}$ for the ∞-category generated by a point under finite colimits. We also consider $\mathsf{S}_{n-\mathrm{fin}}$ and $\mathsf{S}_{\pi-\mathrm{fin}}$, the full subcategories of n- and π-finite spaces.

Homotopy cardinality defines a functor[2] $\mathsf{S}_{\pi-\mathrm{fin}} \to \mathbb{Q}_{\ge 0}$. It is the unique extension of the cardinality of finite sets that is homotopy invariant, additive w.r.t. disjoint unions, and multiplicative w.r.t. fibrations. See the answer to my MO question.

[2] I don't think this is actually a functor. It is a function from the set of equivalence classes of π-finite Kan complexes to $\mathbb{Q}_{\ge 0}$.

I.2 (1/4) Colimit completions and filtering classes

Here's a story I really like. Consider an ordinary category C. Its presheaf category $\mathrm{PShv}(\mathsf{C}) := \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ is the "free completion at colimits" of C in the following sense.

Theorem I.1. The Yoneda embedding $\mathcal{Y} : \mathsf{C} \to \mathrm{PShv}(\mathsf{C})$ is a free cocompletion of C. That is, $\mathrm{PShv}(\mathsf{C})$ has small colimits, and if D admits small colimits, then restriction along $\mathcal{Y}$ defines a natural equivalence

$$\mathcal{Y}^* : \mathrm{Fun}^{cocts}(\mathrm{PShv}(\mathsf{C}), \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathsf{C}, \mathsf{D}).$$

This is standard. I think it also characterizes $\mathcal{Y}$ (and hence $\mathrm{PShv}(\mathsf{C})$) by the usual "thing satisfying universal property is unique" argument. Here is another characterization.

Definition I.3. Consider a presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. Its *category of elements* $\mathrm{el}\, F$ has (objects) transformations $\mathcal{Y}c \to F$ and (arrows) transformations $\mathcal{Y}c \to \mathcal{Y}d$ such that the evident triangle commutes.

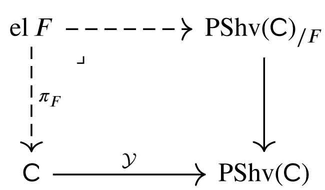
Proposition I.1 (Density)**.** There is a canonical map $\mathrm{el}\, F \to \mathsf{C}$. Its colimit canonically presents F, i.e.,

$$\mathrm{colim}(\mathrm{el}\, F \to \mathsf{C} \to \mathrm{PShv}(\mathsf{C})) \cong F.$$

The density theorem canonically associates to each presheaf $F : * \to \mathrm{PShv}(\mathsf{C})$ a diagram of representables $\mathrm{el}\, F \to \mathrm{PShv}(\mathsf{C})$, which we equivalently regard as its underlying diagram $\mathrm{el}\, F \to \mathsf{C}$. I think you can upgrade this to say "the category $\mathrm{PShv}(\mathsf{C})$ is *equivalent* to the category of diagrams in C."

We have roughly provided three equivalent definitions of $\mathrm{PShv}(\mathsf{C})$: it is (I) the functor category $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$, (II) the category of diagrams in C,[3] and (III) the free colimit completion of C. The equivalence (I) → (III) associates to F its diagram $\mathrm{el}\, F \to \mathsf{C}$. The equivalence (III) → (I) takes the colimit of the diagram. Either can be shown to fit the description (II).

Remark I.6. For a presheaf F, its category of elements can also be defined as the pullback below. The map $\mathrm{el}\, F \to \mathsf{C}$ is again evident. Perhaps because of this definition, we sometimes write $\mathsf{C}/F := \mathrm{el}\, F$.



Remark I.7. The category of elements C/F can also be defined as the *comma category* $* \downarrow F$. I honestly do not know about comma categories and am not sure I need to know about them right now.

Often, we want to think about the completion C at a *certain class* of colimits—say, filtered or sifted colimits. We described the colimit completion $\mathrm{PShv}(\mathsf{C})$ in three ways (I), (II), and (III) above; these suggest three ways to complete at a *chosen class* of colimits. Namely, the completion of C at a "nice" colimits should be...

(I) A subcategory of "nice" presheaves in $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$;

(II) A subcategory of "nice" diagrams in C; and

(III) The "free nice colimit completion" of C (in the sense of a universal property).

All three models are useful. For example, *filtered colimits* are easiest to define as colimits over *filtered diagrams*, and this diagrammatic definition suggests we define the *ind-completion* $\mathrm{Ind}(\mathsf{C})$[4] as the category

[3] The description (III) needs more attention than we have provided. First of all, you want *small* diagrams. Then you must define morphisms. This requires some care. Will I get around to typing this out?

[4] Filtered colimits used to be called inductive limits, so we call the completion of C at filtered colimits its "ind-completion" or "indization."

of filtered diagrams in C, in the likeness of (II).[5] (They are equivalently the diagrams whose colimits in Set commute with finite limits.) Meanwhile, *sifted colimits* are not so easily described by the shape[6] of their diagram; we say a diagram is *sifted* if its colimits commute with finite products (in Set). This suggests a definition of the sifted completion in the likeness of (I): when C has finite (co?)products, I think you can define its sifted completion as the full subcategory of presheaves commuting with finite products. (You can also define it as the subcategory of sifted diagrams in C, but I'm trying to make the point that (I) and (II) are both natural.)

Remark I.8 (Further reading)**.** I just said lots of stuff, but maybe left out lots of stuff. A little while ago, I gave a detailed account of the above picture in Lecture 6 of our condensed seminar, notes here.

I have not really thought about colimit completions for ∞-categories before. But, it seems the ordinary picture persists verbatim. Let me spell out an extended example, the motivating example in Charles' paper [Rez22], which is what made me start thinking about colimit completions again.

Example I.3 (Completion at κ-filtered colimits)**.** For a regular cardinal κ, an ∞-category C is called *κ-filtered* if every κ-small simplical set diagram $K \to \mathsf{C}$ admits an extension $K^{\triangleright} \to \mathsf{C}$ [Lur08, 5.3.1.7]. Parallel to the ordinary case, κ-filteredness is equivalent to commutation with all small limits in Space [Lur08, 5.3.3.3]. It turns out that κ-smallness is a generally well-behaved property (e.g., it is preserved by categorical equivalences), enough so that we may *complete* C *at κ-small colimits* to obtain $\mathrm{Ind}_\kappa(\mathsf{C})$. Also parallel to the ordinary picture, $\mathrm{Ind}_\kappa(\mathsf{C})$ admits models in the likeness of (I), (II), and (III) above:

(I) In terms of presheaves, $\mathrm{Ind}_\kappa(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by filtered colimits of representables [Lur08, 5.3.5.4]. Furthermore, if C admits small colimits, then Ind_κ can be more concretely described as the full subcategory spanned by preseheaves preserving finite limits.

(II) In terms of point categories, $\mathrm{Ind}_\kappa(\mathsf{C})$ consists of diagrams $\mathsf{J} \to \mathsf{C}$ such that J is a κ-filtered simplicial set(?)

(III) In terms of its universal property, $\mathrm{Ind}_\kappa(\mathsf{C})$ admits κ-filtered colimits, Yoneda factors as $\mathcal{Y} : \mathsf{C} \to \mathrm{Ind}_\kappa(\mathsf{C})$, and if D admits κ-filtered colimits, then restriction along $\mathcal{Y}$ defines an equivalence

$$\mathrm{Fun}_\kappa(\mathrm{PShv}(\mathsf{C}), \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathsf{C}, \mathsf{D}).$$

In other words, if D admits κ-filtered colimits, then each arrow $\mathsf{C} \to \mathsf{D}$ admits an essentially unique extension to $\mathrm{Ind}_\kappa(\mathsf{C})$ [Lur08, 5.3.5.10].

Remark I.9. Lurie's verbatim definition [Lur08, 5.3.5.1] is "$\mathrm{Ind}_\kappa(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by presheaves F which classify right fibrations $\mathsf{C}/F \to \mathsf{C}$ such that C/F is κ-filtered." I *feel* like this is just (I) and (II) above at the same time. Lurie basicallly says in [Lur08, 5.3] that he exhibits model (II), but he does not seem to exhibit that explicitly. But I actually do not know anything about (un)straightening, so I cannot really navigate.

Remark I.10. In the ordinary case, it is standard to characterize (say) ind-objects by their point category, i.e., to say that an ind-object is a filtered diagram in C, equivalently a presheaf $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ whose category of elements is filtered (and cofinal) [KS06, 3.3.13, 6.1.5].

I regard Lurie's definition of $\mathrm{Ind}_\kappa(\mathsf{C})$ as an amalgam of models (I) and (II). These are "concrete" models which one shows have the relevant universal property. Charles' paper [Rez22] considers the general question:

[5] Then you can recharacterize it in the likeness of (I) and (III). For (I), it turns out $\mathrm{Ind}(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by filtered colimits of representable presheaves. If C is finitely cocomplete, so that C^{op} has finite limits, then we can be more explicit: $\mathrm{Ind}(\mathsf{C})$ is precisely those presheaves commuting with finite colimits. For (III), take the universal property of $\mathrm{PShv}(\mathsf{C})$ but replace "cocontinuous" with "preserves filtered colimits."

[6] I think there is such a description, but it is not nice.

for which classes of ∞-categories $\mathcal{F}$ do these "concrete" models still work (i.e., actually have the relevant universal property)? More precisely, which classes of small ∞-categories $\mathcal{F}$ are such that for every ∞-category C, the category $\{X \in \mathrm{PShv}(\mathsf{C}) : \mathsf{C}/X \in \mathcal{F}\}$ has the "free $\mathcal{F}$-colimit completion" property?

For this purpose, we make two definitions. First, we define $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) := \{X \in \mathrm{PShv}(\mathsf{C}) : \mathsf{C}/X \in \mathcal{F}\}$. Second, we define $\mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ as the *minimal full subcategory of* $\mathrm{PShv}(\mathsf{C})$ *generated by representables under $\mathcal{F}$-colimits,* to mean the smallest full subcategory containing $\mathcal{Y}(\mathsf{C}) \subseteq \mathrm{PShv}(\mathsf{C})$ and closed under $\mathcal{F}$-colimits. Yoneda factors as $\mathcal{Y} : \mathsf{C} \to \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$, and this functor is a *free $\mathcal{F}$-colimit completion* [Lur08, 5.3.6.2]. Then we seek to compare $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ and $\mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$.

This is not complicated. First, note that $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ since any presheaf $X \in \mathrm{PShv}(\mathsf{C})$ is a colimit of $\mathsf{C}/X \to \mathsf{C} \to \mathrm{PShv}(\mathsf{C})$, thus X is an $\mathcal{F}$-colimit of representables as soon as $\mathsf{C}/X \in \mathcal{F}$. Now, if we can show that $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}(\mathsf{C})$ contains $\mathcal{Y}\mathsf{C}$ and is stable under $\mathcal{F}$-colimits, then we get the reverse inclusion are conclude

$$\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) = \mathrm{PShv}_{\mathcal{F}}(\mathsf{C}).$$

We consider sufficient conditions for $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ to obtain these properties.

- For which $\mathcal{F}$ does $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ always contain the representables? For an object c, we have $\mathsf{C}/\mathcal{Y}c \cong \mathsf{C}_{/c}$. Hence, you could ask that $\mathcal{F}$ contains all ∞-categories with a terminal object. Alternatively(?): there c is terminal, so $\mathcal{Y}c$ is a terminal presheaf. By definition, we in fact have $\mathcal{Y}c \in \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$. Hence, if $\mathcal{F}$ contains D whenever $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ contains a terminal presheaf, then we would have $\mathcal{Y}c \in \mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$.
- I have nothing to say about when $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ is stable under $\mathcal{F}$-colimits. See the proof of Prop. 4.2 in Charles paper [Rez22].

We say that $\mathcal{F}$ is *filtering* if it contains D whenever $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ contains a terminal presheaf. We seek this property (rather than just ask that $\mathcal{F}$ contain D whenever D has a terminal object) because filtration turns out to be closely related to our problem:

Proposition I.2. Suppose that a family of ∞-categories $\mathcal{F}$ is such that for any C, the full subcategory inclusion $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ is an isomorphism. Then $\mathcal{F}$ is filtering.

Proof. If $1 \in \mathrm{PShv}(\mathsf{D})$ denotes a terminal presheaf, then $\mathsf{D}/1 \cong \mathsf{D}$. Therefore[7] $\mathsf{D}/1 \cong \mathsf{D}$ lies in $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ and thus $\mathrm{Ind}_{\mathcal{F}}(\mathsf{D})$ by assumption. This exactly says that $\mathcal{F}$ is filtering. □

To summarize, we explained that if $\mathcal{F}$ is filtering, then $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ is an equivalence, and we proved the converse. Hence, given a class of ∞-categories $\mathcal{F}$, the "diagrammatic" model for the $\mathcal{F}$-colimit completion $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ (presheaves whose point category belong to $\mathcal{F}$) is an actual $\mathcal{F}$-colimit completion (in the sense of possessing the relevant universal property, equivalently $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \hookrightarrow \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ being an equality) if and only if $\mathcal{F}$ is filtering. I wonder what the diagrammatic model is good for in general. Charles also studies the notion of filtering classes, which I have not read yet. May also be interesting to look at [Du23].

Remark I.11. In light of this writing, my mind has changed to consider there as being only *two* truly distinct models: a *diagrammatic* model and a *universal property* model. Maybe it's not right to even call the latter a model, since it's really what we're trying *to model*, but that seems to be the language I've found most efficient.

Remark I.12. Someone should rename *filtering classes*. But I am not sure what a good name would be. I do not like *filtering* because we already have *filtered/filtering/filtrant* colimits, which are a very specific example of filtering classes. Something to reflect that $\mathcal{F}$-diagrams are special. Diagrammatic? Graphical? Figurative? "Graphical classes" has a nice ring to it. Or just call them "graphics." Or "complete graphics."

[7] Hmm. Why does $\mathsf{D}/1$ lie in $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$? Does

I.3 (1/15) Reminding myself what presentability, dualizability, and stability are

Classes just started at UIUC, but I am in Chicago while Ishan and Efimov are giving lectures. Today I want to define a category $\mathsf{Pr}^{\text{dual}}_{whichcontainsthecategory}\mathsf{Cat}^{\text{perf}}_{\infty} = \{\text{small, stable, idempotent-complete } \infty\text{-categories}\}$ and out of which we can define *continuous ("Efimov") K-theory* $\mathsf{Pr}^{\text{dual}}_{\mathsf{Sp}}$, extending the algebraic K-theory. I have not actually thought about presentability or dualizability before, nor how they interact with stability. I'm going to review these quickly. Here are some things I am looking at:

- Peter Haine, *Descent for sheaves on compact Hausdorff spaces* here.
- Hoyois' continuous K-theory notes here.
- Parts of Mortiz Groth's notes here, particularly the bits about presentability.
- Dustin Clausen's lectures about Efimov K-theory (on Youtube).
- He Li's Efimov K-theory notes here.
- Chapter 5 of HTT [Lur08].
- Alberto Garcìa-Raboso's notes on stable ∞-categories.
- Yonatan's notes on stable ∞-categories.

First I want to remember what presentability is. In nice cases, colimits are like unions. For instance, a group is the colimit of its directed poset of subgroups. In fact, you can just take that of its *finitely generated* subgroups. We can actually make a categorical equivalence

$$\mathrm{Ind}(\mathsf{Ab}_{\mathrm{fg}}) \cong \mathsf{Ab},$$

the point being that although Ab is large, it admits a small subcategory of ind-generators. This should be considered an essential detail of the structure of Ab, and it is both practically and philosophically important.

Toward pinning down this "big thing is secretly small and this helps us to work with it" idea, consider the *adjoint functor theorem*: a functor $F : \mathsf{A} \to \mathsf{B}$ between cocomplete categories admits a right adjoint if and only if F preserves colimits and satisfies the *solution set condition*. Without going into details, that condition is a certain smallness condition—then is reasonable to think that it is automatically fulfilled whenever A, B are themselves "small enough."

Definition I.4. Say that an ∞-category C is *presentable* if it is cocomplete and *accessible*: there exists a regular cardinal κ such that C has κ-filtered colimits and admits a small subcategory C_0 such that $\mathrm{Ind}_\kappa(\mathsf{C}_0) \cong \mathsf{C}$.

Theorem I.2 (Generalized adjoint functor theorem)**.** A functor between presentable, cocomplete ∞-categories admits a right adjoint if and only if it preserves colimits.

Remark I.13. Since $\mathcal{Y} : \mathsf{D} \to \mathrm{Ind}(\mathsf{D})$ is fully faithful, accessibility amounts to the admittance of a small subcategory of ind-generators. Also, presentable categories admit *bilimits*.

Remark I.14. See [Lur08, §5.4] for a detailed review of accessbility.

Presentability is not a rare property, and it is generally well-behaved. It can be characterized in terms of presheaves:

Theorem I.3. An ∞-category C is presentable if and only if C is an *accessible localization* of $\mathrm{PShv}(\mathsf{D})$ for some small D: there exists a functor $F : \mathrm{PShv}(\mathsf{D}) \to \mathsf{C}$ such that (loc.) F admits a fully faithful right adjoint and (acs.) F commutes with all κ-filtered colimits for some regular κ.

I feel as if I should have something to say about this, but I do not. An idea: let D be small and regard the free colimit completion $\mathrm{PShv}(\mathsf{D})$ as a a "free, small collection of generators." Next, think of an accessible localization $\mathrm{PShv}(\mathsf{D}) \to \mathsf{C}$ as "imposing a small number of relations." Then our characterization seems natural: presentable categories are those with a "small" presentation by generators and relations. It is unclear to me how far one can expound this idea, what the role of accessibility is for the localization insofar as it makes this idea work, ... I do not want to waste time on these details right now. Return to this.

Definition I.5. We organize presentable ∞-categories into an ∞-category. Its objects are (not necessarily small) presentable ∞-categories and its morphisms are the cocontinuous functors [Lur08, 5.5.3.1]. By the adjoint functor theorem, we can also say this category has left adjoints as morphisms. We write Pr^L for this category.

Misc properties.

Proposition I.3 ([Lur08, 5.5.3.8])**.** If C, D are presentable, then the full subcategory $\mathrm{Fun}^L(\mathsf{C}, \mathsf{D}) \subset \mathrm{Fun}(\mathsf{C}, \mathsf{D})$ spanned by left adjoints is presentable. (In fact, C can be any simplicial set.)

Proposition I.4. If C, D are presentable, then there exists a presentable category $\mathsf{C} \otimes \mathsf{D}$ which is the universal recipient of a functor from $\mathsf{C} \times \mathsf{D}$ that is colimit-preserving in both variables seperately. One presentation is $\mathsf{C} \otimes \mathsf{D} \cong \mathrm{Fun}^{cts}(\mathsf{C}^{\mathrm{op}}, \mathsf{D})$. It inherits this (symmetric monoidal) tensor product as a full subcategory (on the presentables) of $\hat{\mathsf{Cat}}_\infty(K)$, the ∞-category of ∞-categories with small colimits and colimit-preserving functors. (Which has the Lurie tensor product?) Maybe c.f. [Lur17, 4.8.1.5]. See also nLab.

Ok, now dualizability. In a monoidal category C, an object $x \in \mathsf{C}$ is called *dualizable* if (existence of "dual" with co/evaluation maps). We consider this in the case of vector spaces.

Example I.4. Consider a vector space $V \in \mathsf{Vect}_k$. A candidate for its dual is $V^* = \mathrm{Hom}(V, k)$ and there is an obvious map $\mathrm{ev} : V \otimes V^* \to k$. We would like a map $\mathrm{coev} : k \to V \otimes V^*$, which amounts to the choice of an element $v \in V \otimes V^*$. If $(e_i)_I$ is a basis for V, we have coordinates $v = \sum c_{i,j} e_i \otimes e_j^*$ such that only finitely many $c_{i,j}$ are nonzero. The axioms for co/evaluation imply that for each $i \in I$, some $c_{i,j}$ is nonzero, hence I must be finite, i.e., V must be finite-dimensional. Conversely, if V is finite-dimensional, then it is dualizable with dual V^*, in fact we may identify $V \otimes V^* \cong \mathrm{End}(V)$ (which requires a choice of basis) and define coev by $1_k \mapsto \mathrm{id}_V$.

As this suggests, dualizability is a sort of finiteness condition. But, apparently different from that of presentability. Dualizable objects inherit some of the theory of finite-dimensional vector spaces, for instance a notion of *traces of endomorphisms* and *dimension* (the trace of the identity).

Remark I.15. I think that if C is monoidal with internal homs, then an object $X \in \mathsf{C}$ is dualizable if and only if the canonical pairing $X \otimes \mathrm{Hom}(X, 1) \to \mathrm{End}(X)$ is an isomorphism. The category Vect_k has internal homs, and we saw that the map is an isomorphism therein, so this checks out. Also in Vect_k, we know that for non-dualizable (equivalently, infinite dimensional) V, that $V \otimes V^*$ is nicer than $\mathrm{End}(V)$ in general. In other monoidal categories *without* internal homs, I wonder if you can ever treat $X \otimes X^*$ like a well-behaved substitute for $\mathrm{End}(X)$. I had this thought during coffee with Anthony and Sam and they asked, "have you heard of a star-autonomous category?"

We want to think about dualizability in the full subcategory $\mathsf{Pr}_{\mathsf{Pr}^L}$ spanned by the presentable, stable ∞-categories. A good question is, why are we making the stability hypothesis?

Recall that an ∞-category is *stable* if it has a zero object, has fibers and cofibers, and fiber sequences coincide with cofiber sequences. I can weakly explain where this comes from: even in classical homotopy theory, we care deeply about pushouts, pullbacks, and (co)fibrations. These are tricky notions when you work with spaces (not intractable), partly due to the rebellious nature of homotopy (co)limits in the classical stable homotopy category. This gives some impetus for *stable* homotopy theory: stable phenomena somehow simplify the story. I wish I could give a concrete, classical example of this, but all I can think of is "fibrations and cofibrations of spectra coincide." Maybe some helpful discussion here and here.[8]

Remark I.16. A functor between stable categories is called *reduced* if it preserves zero, and *exact* if in addition it preserves (co)fiber sequences. Since the zero object and (co)fibration sequences are (co)limits, it is preserved by left adjoints, hence these properties are superfluous in Pr^L and Pr^R.

Given a pointed category C, it has a *suspension functor* Σ_{C} once it has cofibers, and it has a *loop functor* Ω_{C} once it has fibers. Stability is characterized by either suspension or looping being an equivalence. This presents an idea: does formally inverting Σ_{C} or Ω_{C} present a "stabilization"? Note that Cat^{ex} occurs as a full subcategory of $\mathsf{Cat}_*^{\text{fincolim}}$ (resp. $\mathsf{Cat}_*^{\text{finlim}}$), spanned by those categories whose suspension (resp. loop) functor is an equivalence. "Stabilization" should mean an adjoint to these inclusions.

These inclusions in fact have left and right adjoints, and we get the left adjoints in the manner described.

Yonatan's notes have a bit about suspension and the triangulated structure for stable categories. Maybe read this later.

Proposition I.5. Let C be pointed. If C has finite colimits (cofibers and suspension in particular), then the colimit $\mathsf{Sp}^{\Sigma}(\mathsf{C}) := \operatorname{colim}(\mathsf{C} \xrightarrow{\Sigma} \mathsf{C} \to \cdots) \in \mathsf{Cat}_*^{\text{fincolim}}$ is stable, and this extends to a left adjoint to the inclusion $\mathsf{Cat}^{\text{ex}} \hookrightarrow \mathsf{Cat}_*^{\text{fincolim}}$. Dually, if C has finite limits, we get a left adjoint to $\mathsf{Cat}^{\text{ex}} \hookrightarrow \mathsf{Cat}_*^{\text{finlim}}$ given by $\mathsf{C} \mapsto \mathsf{Sp}^{\Omega}(\mathsf{C})$.

Rather strangely(?), even if C is (finitely) complete and cocomplete, their suspension-spectra and loop-spectra are not generally equivalent. That is, we do not always have $\mathsf{Sp}^{\Sigma}(\mathsf{C}) \simeq \mathsf{Sp}^{\Omega}(\mathsf{C})$. For example, finite spectra and Ω-spectra do not coincide. But...

ponder

Observation I.2. Presentability puts us somewhere nice: we get *all* limits and colimits, for instance. Also, the natural notion of morphisms of presentable categories (limit or colimit preserving, you choose) already fits that for categories we usually consider stabilizing (finite limit or colimit preserving, you choose). And if your presentable categories were already stable, either notion is automatically an exact functor.

Hence, at least formally, it seems easy and convenient to consider *presentable* stable ∞-categories. Since presentable demands (co)limits, the characterization of stability simplifies: *if C is a pointed, presentable ∞-category, then $\Sigma_{\mathsf{C}}, \Omega_{\mathsf{C}}$ are defined, and C is stable iff either of these is an equivalence.*

(Stuff; universal "stabilization" property; the realization of Ω-spectra as the finite shifts of finite suspension spectra; the definition of the ∞-category of spectra; the definition of its symmetric monoidal *smash product*!)

Probably give this it's own day.

I.4 (1/24) Stabilization, the ∞-category of spectra, and the smash product

My notes previously got me thinking about stability, and then Charles told me some things about presentability and the smash product on the ∞-category of spectra, so now I am going to think a bit about all that. I have already defined *stable ∞-categories* at least twice, which behave "like spectra," or maybe "like chain complexes of abelian groups." Last time, I also thought a bit about how stability simplifies in the presence of presentability (although it seems I didn't actually write about that). This makes sense, since presentability forces existence of (co)limits, and stability cares about cetain (co)limits (and needs them for the essential Σ and Ω). But that does not do justice to the fun which presentability brings to the party. I'm going to be primarily reading Groth's notes.

For a pointed category C, recall the definition of *triangles*: they are the composable pairs of morphisms (g, f) together with 2-cells realizing $gf \simeq h$ and $h \simeq 0_{X,Y}$. These form a category, namely the full subcategory of $\operatorname{Fun}(\Delta^1 \times \Delta^1, \mathsf{C})$ spanned by functors mapping the "bottom-left" vertex to 0. Consider that $\Delta^1 \times \Delta^1$ is both a left and right cone [Lur24, Tag 0165]:

$$(\Lambda_0^2)^{\triangleleft} \cong \Delta^1 \times \Delta^1 \cong (\Lambda_2^2)^{\triangleright}.$$

We say a triangle is *exact* if it is a limit as a left cone, and *coexact* if it is a colimit as a right cone. If C admits all finite (co)limits, we denote by

$$\mathsf{C}^{\Sigma} \subseteq \operatorname{Fun}(\Delta^1 \times \Delta^1, \mathsf{C}) \quad \text{and} \quad \mathsf{C}^{\Omega} \subseteq \operatorname{Fun}(\Delta^1 \times \Delta^1, \mathsf{C})$$

the full subcategories spanned by coexact and exact triangles *with the bottom-left and top-right corners zero*, respectively. To form a suspension ΣX, no data is needed beyond specifying X; diagramatically, a functor $F \in \mathsf{C}^{\Sigma}$ should be determined by its top-left corner. Dually for ΩX. Abstact nonsense says that indeed, $\operatorname{ev}_{(0,0)} : \mathsf{C}^{\Sigma} \to \mathsf{C}$ and $\operatorname{ev}_{(1,1)} : \mathsf{C}^{\Omega} \to \mathsf{C}$ are acyclic Kan fibrations.

[8] Think about these. Maybe think about connectivity results.

Definition I.6. Suppose that C is pointed and admits finite (co)limits. Since $\mathrm{ev}_{0,0} : \mathsf{C}^{\Sigma} \to \mathsf{C}$ is acyclic, it admits a section $s_{\Sigma} : \mathsf{C} \to \mathsf{C}^{\Sigma}$, well-defined up to a contractible choice. We can thus define (up to a contractible choice) the *suspension functor* $\Sigma_{\mathsf{C}} = s_{\Sigma} \circ \mathrm{ev}_{(0,0)}$. We define the *loops functor* Ω_{C} identically.

The functors $\Sigma_{\mathsf{C}}, \Omega_{\mathsf{C}}$ are adjoint. Furthermore, if C is stable (i.e., exact triangles = coexact triangles), then they are inverse equivalences. The converse is also true.

We are interested in stabilizing categories. As stability is characterized by Ω_{C} being an equivalence, this means inverting Ω_{C}, i.e. by taking the colimit of $\cdots \to \mathsf{C} \to \mathsf{C}$. That is categorical. But there's an analogy here with algebraic topology, and we wonder about forming "spectrum objects" in C. In fact, we can do this, giving a more explicit model for the stabilization. We can apply this in the case of spaces—the categorical properties of the stabilization will give us the *symmetric monoidal smash product* of spectra, and the description as "spectrum objects" will make clear that we are talking *actual* spectra, the kind we care about from classical algebraic topology.

Let C be pointed and finitely (co)complete. A *prespectrum* in C is a functor $X : \mathbb{N} \times \mathbb{N} \to \mathsf{C}$ such that $X(i,j) = 0$ whenever $i \neq j$. A prespectrum X determines a sequence of triangles in C, and thus maps

$$\Sigma X_n \to X_{n+1} \quad \text{and} \quad X_n \to \Omega X_{n+1}.$$

We say that X is a *spectrum* if all the maps $X_m \to \Omega X_{m+1}$ are equivalences. We say that X is a *spectrum below n* if that is true for $m < n$. The following theorem says that (with mild hypotheses) spectrum objects model the stabilization.

Theorem I.4 (DAG I, 8.14)**.** For an arbitrary ∞-category C, we define its *stabilization* $\mathrm{Stab}(\mathsf{C}) := \mathsf{Sp}(\mathsf{C}_*)$. Note that if C is pointed, then $\mathsf{C}_* \to \mathsf{C}$ is a trivial Kan fibration, and so is the induced $\mathrm{Stab}(\mathsf{C}) \to \mathsf{Sp}(\mathsf{C})$. If C is pointed with finite limits, then

$$\mathsf{Sp}(\mathsf{C}) \cong \mathrm{colim}(\cdots \xrightarrow{\Omega} \mathsf{C} \xrightarrow{\Omega} \mathsf{C}).$$

Remark I.17. This presentation follows Moritz's notes and Lurie's DAG. However, Lurie's presentation in HA is different. There, he wants to faithfully reenact the story of *infinite loop spaces* (in particular, Brown representability and excision for cohomology) in the setting of ∞-categories. For C with finite limits, he defines $\mathsf{Sp}(\mathsf{C}) := \mathrm{Fun}(\S^{\mathrm{fin}}_*, \mathsf{C}) \subseteq \mathrm{Fun}(\dots)$ the full subcategory of reduced, excisive functors [Lur17, 1.4.2.8]. These approaches are equivalent. Given such a functor X, one may consider $X_n := X(S^n)$, and we get equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$ by considering excision for the following pushout diagram.

$$\begin{array}{ccc} S^n & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & S^{n+1} \end{array}$$

There are forgetful functors $\mathsf{Sp}(\mathsf{C}) \to \mathsf{Sp}_{\leq n}(\mathsf{C}) \to \mathsf{PSp}(\mathsf{C})$. We can ask about adjoints. Let's first ask about finding $\mathsf{PSp}(\mathsf{C}) \to \mathsf{Sp}_{\leq n}(\mathsf{C})$. Given a prespectrum X, one may imagine that "$L_n(X) :=$ the free Ω-spectrum below n on X" should be $X[k]$ in degrees $k \geq n$ and should be $\Omega^k X[n]$ in degrees $k < n$. Heuristically, $L_n(X)$ should be determined by the part of X in degrees $\geq n$, obtained by just chopping off lower degrees and refilling them by looping downward.

Lurie spells all this out in [Lur09, §8] (and these approximations $L_n X$ are needed to prove 8.14 above). The basic idea is as follows. For $-\infty \leq a \leq b \leq \infty$, we define

$$Q(a,b) := \{(i,j) : i \neq j \text{ or } a \leq i = j \leq b\} \subseteq \mathbb{N} \times \mathbb{N}.$$

We want to define $L_n(X) := \mathrm{Ran}_{Q(n,\infty) \hookrightarrow \mathbb{N}\times\mathbb{N}}(X|_{Q(n,\infty)})$. Since that inclusion $Q(n,\infty) \hookrightarrow \mathbb{N} \times \mathbb{N}$ is quite "finite," this is possible as soon as C has finite limits.

Proposition I.6 ([Lur09, 8.12])**.** C pointed with finite limits, $X_0 \in \mathsf{PSp}^{\infty}_{a}(\mathsf{C})$. Then

(1) There exists $X \in PSp^{\infty}_{a-1}(\mathsf{C})$ which is a right Kan extension of X_0.
(2) There exists $X \in PSp^{\infty}_{-\infty}(\mathsf{C})$ which is a right Kan extension of X_0.
(3) An object $X \in PSp^{\infty}_{-\infty}(\mathsf{C})$ is a right Kan extension of X_0 if and only if X is a spectrum below a.

Remark I.18. Lurie also states a characterization of $X \in PSp^{\infty}_{a-1}(\mathsf{C})$ right Kan-extending X_0, except there seems to be a typo that renders it unclear.

Hence, for C with finite limits, we have described a sequence of functors $\mathrm{id} \to L_0 \to L_1 \to \cdots$ such that

(1) $L_n X$ is a spectrum below n,
(2) For $m \geq n$, the map $X[m] \to L_n X[m]$ is an equivalence,
(3) If X is already a spectrum below n, then the map $X \to L_n X$ is an equivalence, and
(4) As a functor $\mathsf{PSp}(\mathsf{C}) \to \mathsf{PSp}_{\leq n}(\mathsf{C})$, each L_n is left-adjoint to the inclusion $\mathsf{PSp}_{\leq n}(\mathsf{C}) \hookrightarrow \mathsf{PSp}(\mathsf{C})$.

Properties (1) and (2) are immediate if you unwind everything. Property (3) follows from (1) and (2). Property (4) is not hard either. Next, toward an adjoint $\mathsf{PSp}(\mathsf{C}) \to \mathsf{Sp}(n)$, it is natural to ask about the colimit of this tower of approximations L_n. This works under some mild hypotheses.

Proposition I.7. If C is pointed and admits finite limits and countable colimits, and $\Omega_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$ preserves sequential colimits, then $L := \mathrm{colim} L_n : \mathsf{PSp}(\mathsf{C}) \to \mathsf{PSp}(\mathsf{C})$ is a localization with essential image $\mathsf{Sp}(\mathsf{C})$. Under these conditions, we call L a *spectrafication functor*.

Finish this

I.5 (1/29) Localizations I

I secretly never learned anything, ever. This includes most of algebraic topology—somehow, it was my third(?) undergrad course, so I spent that semester learning mathematical maturity, not actual algebraic topology. Rather than patiently fill the holes in my knowledge, I moved forward to homotopy theory, at which point I was mature enough to actually learn homotopy theory, although that was (and still is) complicated since my algebraic topology is lacking. The water settles and some holes get filled idly, but maybe not all of them. This is most apparent when I try to do chromatic things. All that is to say, today I want to review localizations, with an eye toward localizations at spectra (homology theories), and I am going to start from basics. Some references are:

- Tyler Lawson's expository article about Bousfield localization [Law20].
- nLab.
- Ishan's notes.
- Paul VanKoughnett's thesis here.

Fix a category C and a class $S \subseteq \mathrm{Mor}(\mathsf{C})$ of morphisms. We say that an object Z is *S-local* if for each $s \in S$, the pullback $s^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an equivalence. We say that a morphism $f : X \to Y$ is an *S-equivalence* if for each S-local Z, the pullback $f^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an equivalence. (Hence, each $s \in S$ is an S-equivalence.) We call a morphism $L : X \to Z$ an *S-localization* if it is an S-equivalence and Z is S-local.

Proposition I.8. If two S-localizations of an object X exist, then they are isomorphic under X.

Proof. Given two S-localizations $Z \xleftarrow{L} X \xrightarrow{K} Z'$. Since Z, Z' are S-local, their represented functors invert S-equivalences. In particular, we have bijections $K^* : \mathrm{Hom}(Z', Z) \cong \mathrm{Hom}(X, Z)$ and $L^* : \mathrm{Hom}(Z, Z') \cong \mathrm{Hom}(X, Z')$. So we may consider $(L^*)^{-1}K : Z \leftrightarrow Z' : (K^*)^{-1}L$. Pullback is just precomposition, whence the relevant triangles commute, so that these are *morphisms under* X. Furthermore, uniqueness implies that these two morphisms are inverse equivalences. □

Hence we speak of *the* S-localization of X, written $L^S X$ or just LX (the issue of existence notwithstanding). Assuming existence, the localization morphisms $X \to LX$ are functorial. Here's that formally:

Proposition I.9. Consider $\mathrm{Loc}^S(\mathsf{C})$ the category of S-localizing morphisms and commutative squares between them. The forgetful functor $\mathrm{Loc}^S(\mathsf{C}) \to \mathsf{C}$ given by $(X \to LX) \mapsto X$ is fully faithful.

In particular, if C has S-localizations, then $\mathrm{Loc}^S(\mathsf{C}) \to \mathsf{C}$ is an equivalence. In that case we may choose an inverse $\mathsf{C} \to \mathrm{Loc}^S(X)$ written as $X \mapsto (X \to LX)$, and forgetting gives us a *functorial localization functor*

$$\mathsf{C} \to L^S\mathsf{C}.$$

Proposition I.10. If C has S-localizations, then we have described a functorial choice of localization $X \mapsto LX$ which is left adjoint to the forgetful functor $LX \mapsto X$.

Some examples.

Example I.5. Consider Mon and the inclusion $f : \mathbb{N} \hookrightarrow \mathbb{Z}$. For a monoid M to be f-local, every map $\mathbb{N} \to M$ must extend (uniquely) to a map $\mathbb{Z} \to M$. In other words, each element of M must admit a (unique) inverse. (I parenthesize uniqueness of inverses here because that property is superfluous in monoids.) Hence,

$$L^{\mathbb{N} \hookrightarrow \mathbb{Z}}\mathsf{Mon} = \mathsf{Gp}.$$

Furthermore, group completion $M \mapsto M^{gc}$ presents the localization functor.

Example I.6. Consider Gp and the abelianization map $f : F_2 \to \mathbb{Z}^2$ given by $(1,1) \mapsto (1,1)$. A group G is f-local if every $F_2 \to G$ factors (uniquely) through $\mathbb{Z}^2$. This means that for each $x, y \in G$, the commutator $[x, y]$ vanishes, which occurs precisely when G is abelian. Therefore,

$$L^{F_2 \to \mathbb{Z}^2}\mathsf{Gp} = \mathsf{Ab},$$

and abelianization $G \mapsto G^{ab} = G/[G,G]$ presents the localization functor.

Example I.7. I think that $L^{\mathbb{N}^2 \hookrightarrow \mathbb{Z}^2}\mathsf{Mon} = \mathsf{CMon}$.

Example I.8. Consider $\mathsf{Ab}^{\mathrm{fg}} \subset \mathsf{Ab}$ and let $S = \{\mathbb{Z} \xrightarrow{p} \mathbb{Z} : p \text{ is prime}\}$. An abelian group G is S-local if every p is invertible in G, i.e., when G is a rational vector space. (Note that this description is valid in both Ab and $\mathsf{Ab}^{\mathrm{fg}}$ because $\mathsf{Ab}^{\mathrm{fg}}$ is a full subcategory.) Then within $\mathsf{Ab}^{\mathrm{fg}}$, the only S-local object is $G = 0$, whence S-localization is the zero map and all homomorphisms are S-equivalences. However, the S-localizations in Ab are given by rationalization.

Finish. At least get to Lawson's stuff on unstable and stable settings.

II February

II.1 (2/2) Phony multiplication

Some references (in order of discovery as I wrote) are

- Yigal Kamel's talk for the homotopy theory seminar at UIUC. Good talk!
- Thomason's article.
- Segal's paper "Categories and cohomology theories."
- Dmitri Pavlov's MO question about a higher $S^{-1}S$ construction.
- Gurski-Johnson-Osorno's paper "2-CATEGORICAL OPFIBRATIONS, QUILLEN'S THEOREM B, AND $S^{-1}S$."
- Dan Grayson's expository article "Quillen's work on algebraic K-theory."
- Zbigniew Fiedorowicz's proceedings article "The Quillen-Grothendieck Construction and Extensions of Pairings" c.f. [Fie78].
- Clayton Sherman's chapter "Group representations and algebraic K-theory," c.f. [She82], which has some random details worked out I found helpful.
- Daniel Harrer's thesis "Comparison of the Categories of Motives defined by Voevodsky and Nori" available here. This is mostly unrelated, but Harrer mentions that somewhere in motivic cohomology, one needs the swap map in a tensor category to be an equality, otherwise you get a problem analogous to what Thomason finds for algebraic K-theory. Nice
- Baas-Dundas-Rognes "Two-vector bundles and forms of elliptic cohomology" c.f. [BDR03].

Given a monoid M, we may "formally add inverses" to obtain its *group completion* M^{gp}. If M is commutative, then M^{gp} has a universal property: there exists a monoidal map $i : M \to M^{\text{gp}}$ such that for every abelian group A, monoidal maps $M \to A$ extend surjectively and faithfully to homomorphisms $M^{\text{gp}} \to A$. By virtue of universality, this map i is determined up to isomorphism(?) Alternatively, you could characterize $(-)^{\text{gp}}$ as left adjoint to $U : \mathsf{Ab} \to \mathsf{CMon}$, and then a choice of maps $\{M \to M^{\text{gp}}\}_M$ is a presentation of the left adjoint(?)

That's categorical. But we can do this with our hands. Two ways! Let's just take $M = \mathbb{N}$.

Construction II.1. We define $\mathbb{N}^{\text{gp}}$ as the set of symbols $\{m-n : m, n \in \mathbb{N}\} = \mathbb{N}\times\mathbb{N}$ modulo the equivalence relation *generated by*[9] identifying $a - b \sim c - d$ when there exists $k \in \mathbb{N}$ such that $a + k = c$ and $b + k = d$. You can check that $\mathbb{N}^{\text{gp}} \cong \mathbb{Z}$ and that both the maps $\mathbb{N} \to \mathbb{N}^{\text{gp}}$ given by $m \mapsto m - 0$ and $m \mapsto 0 - m$ are group completions.

Construction II.2. We define $\mathbb{N}^{\text{gp}}$ as $\text{FAb}(\mathbb{N})/\langle (m + n) -_F (m +_F n)\rangle$, where by $\text{FAb}(\mathbb{N})$ we mean the free abelian group on the set $\mathbb{N}$, and by $+_F, -_F$ we mean the group operations in $\text{FAb}(\mathbb{N})$.

Both these constructions readily extend to an arbitrary commutative monoid M. Note that Construction II.2 works for noncommutative M. And there's more: if M is a semiring, to mean a monoid with a multiplication, then $(M, +, \times)^{\text{gp}}$ has a canonical and well-defined multiplication $\times_{M^{\text{gp}}}$ given as

$$((a - b), (c - d)) \mapsto (ac + bd, ad + bc).$$

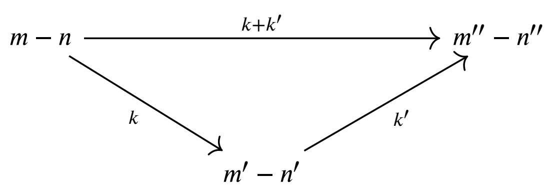
We have described a functor $K_0 : \mathsf{CMon} \to \mathsf{Ab}$. (The second construction works for Mon.) With an eye toward (higher) K-theory, we would like to consider more general input data. A generation of mathematical work (in geometry, topology, number theory, ...) evidences that *many things have K-theory*: (homotopy commutative) rings, spaces, schemes, (various adjectives) categories, ... Moreover, we want highly structured output data, perhaps a *sequence* of abelian groups, or better yet a (nice) K-theory spectrum.

[9] "Generated by" here is necessary since $\sim$ as defined is not reflexive. There are very easy fixes to this, see e.g. the equivalent definitions of the relation on nLab, but this equivalent definition generalizes verbatim.

Often, such things arise by constructing a (category, or higher category/space) C encoding the important information, in such a way that the data unwinds into an infinite loop space structure on the geometric realization $BG = |NC|$.

Let M be a monoid. Following Grayson, we can easily prototype topological models for M^{gp}. For this, we will use simplicial sets, for with simplicial sets we can simultaneously encode the algebra in M and realize it spatially.

Construction II.3. Suppose that M is commutative. We may form the simplicial set $X_0(M)$ wherein vertices are formal differences $m - n$, edges connect $(m + k) - (n + k) \to m - n$ for every $k \in M$, faces fill every composable pair of the form shown below, and so on.



Construction II.4. Do not assume that M is commutative. We may form the simplicial set $X_1(M)$ wherein there is one vertex, an edge for each $m \in M$, a face with edge m, m', and $m + m'$, and so on.

The constructions of $X_0(M)$ and $X_1(M)$ are analogous to Construction II.1 and Construction II.2, respectively. You can check that $\pi_0 X_0(M) \cong M^{\mathrm{gp}}$ and $\pi_1 X_1(M) \cong M^{\mathrm{gp}}$. If M is already a group (resp. abelian group), then $X_1(M)$ (resp. $X_0(M)$) has trivial higher homotopy and models the classifying space BM (resp. ΩBM?). So, we have described two constructions X_0, X_1 which (topologically) model group completion in the sense that $\pi_i X_i(M) \cong M^{\mathrm{gp}}$. The X_1 construction works for noncommutative M, and if M is already group, then $X_0(M) = \Omega BM$ and $X_1(M) = BM$ (in some sense...).

Remark II.1. Group completion also changes homology, see e.g. MO. This persists for categories!

We would like to replace M by a symmetric monoidal category and ask about group completions. We produced $X_0(M)$ by Construction II.1, and this straightforwardly generalizes.

Definition II.1. Suppose that $(\mathsf{C}, \oplus, 0)$ is a symmetric monoidal category. We define its *Quillen completion* $S^{-1}S(\mathsf{C})$ to be the category whose...

- Objects are pairs (A, B).
- Morphisms $(A, B) \to (C, D)$ are triples $(K, A \oplus K \to C, B \oplus K \to D)$, modulo the equivalence relation identifying morphisms when there exists $K \cong K'$ making the obvious four triangles commute.

Remark II.2 (Structure of $S^{-1}S(\mathsf{C})$)**.** Here is some basic structure on the Quillen completion.

- The identity morphism $\mathrm{id}_{(A,B)}$ is $(0, 0 \to A, 0 \to B)$.
- The monoidal structure on C induces one on $S^{-1}S(\mathsf{C})$. It is computed "coordinate-wise," i.e. $(A, B) \oplus (C, D) = (A \oplus C, B \oplus D)$ and likewise for sums of morphisms. If $s_{A \oplus B} : A \oplus B \to B \oplus A$ denotes the swap maps in C, then $(0, s_{AC}, s_{BD}) : (A, B) \oplus (C, D) \to (C, D) \oplus (A, B)$ provide swap maps for $S^{-1}S(\mathsf{C})$.
- There is a *transposition* functor $t : S^{-1}S(\mathsf{C}) \to S^{-1}S(\mathsf{C})$ given by

$$(A, B) \mapsto (B, A) \quad \text{and} \quad (K, \alpha, \beta) \mapsto (K, \beta, \alpha).$$

- There are inclusion functors $i, -i : \mathsf{C} \to S^{-1}S(\mathsf{C})$ given on objects by $i(C) = (0, C)$ and $-i(C) = (C, 0)$. We want to think of the object $(A, B) \in S^{-1}S(\mathsf{C})$ as a formal difference $B - A$,[10] whence this notation makes sense.

We would like to understand $S^{-1}S(\mathsf{C})$ as a "categorical group completion." For this, consider the following three properties that $(\mathsf{C}, \oplus, 0)$ might possess.

(I) C is a groupoid.

(II) C is "cancellative:" For every $A \in \mathsf{C}$, the functor $A \oplus - : \mathsf{C} \to \mathsf{C}$ is faithful.

(III) C has "object-level inverses:" there exists a natural transformation $0 \to \mathrm{id} \oplus t$.

Proposition II.1. If C satisfies (I) and (II) above, then the inclusion $i : \mathsf{C} \to S^{-1}S\mathsf{C}$ given by $A \mapsto (0, A)$ induces a group completion on classifying spaces. That is to say

$$\pi_0(BS^{-1}S(\mathsf{C})) = \pi_0(B\mathsf{C})^{\mathrm{gp}} \quad \text{and} \quad H_0(BS^{-1}S(\mathsf{C})) = H_0(B\mathsf{C})[\pi_0 B\mathsf{C}^{-1}].$$

Suppose that (I) and (II) hold. Since $S^{-1}S\mathsf{C}$ is symmetric monoidal, its classifying space is an H-space, and by the proposition it is an H-group.[11] We regard $(A, B) \in S^{-1}S(\mathsf{C})$ as the formal difference $B - A$, and (A, B) represents this difference in $\pi_0 BS^{-1}S(\mathsf{C})$. Also on π_0, the transposition functor t induces the inverse map. The symmetric monoidal structure gives $BS^{-1}S\mathsf{C}$ its monoidal structure, and we would like to say that t induces its homotopy inverse $g \mapsto g^{-1}$.

Proposition II.2. If C satisfies (I), (II), and (III) above, then the transposition t for the Quillen completion $S^{-1}S(\mathsf{C})$ induces a homotopy inverse for the H-group $BS^{-1}S(\mathsf{C})$.

Proof. Write $Z = BS^{-1}S(\mathsf{C})$. By (III), we may choose a transformation $\eta : 0 \to \mathrm{id} \oplus t$. We get a map $B\eta : B0 \to B(\mathrm{id} \oplus t)$, and this represents a homotopy inducing (in $[Z, Z]$) $0 = [B0] = [B\mathrm{id}] + [Bt] \in [Z, Z]$, which begets $[B\mathrm{id}] = -[Bt]$. Hence, Bt is a homotopy inverse for the H-space Z. □

Thomason's essential observation is that (III) is secretly a very strong condition, unfulfilled in even the most standard cases. For example, neither $\mathsf{C} = \mathsf{Mod}_R^{\mathrm{fg},proj}$ nor its maximal subgroupoid have this property unless $R = 0$. Fortunately, Proposition II.2 still holds when you do not assume (III)! But results implementing the homotopy inverse for $BS^{-1}S(\mathsf{C})$ via t *and utilizing the naturality of* η *in an essential way* still fail (since η does not even exist).

The functor $t \oplus \mathrm{id}$ acts on objects as $(A, B) \mapsto (B \oplus A, A \oplus B)$. Hence, a transformation $\eta : 0 \to t \oplus \mathrm{id}$ amounts to a natural system of morphisms $\eta_{AB} = \{K, 0 \oplus K \to B \oplus A, 0 \oplus K \to A \oplus B\}$. One candidate is

$$\eta_{AB} := \{A \oplus B, s_{A \oplus B}, \mathrm{id}_{A \oplus B}\}.$$

Proposition II.3. The system of morphisms $\{\eta_{AB} : (0, 0) \to [t \oplus \mathrm{id}](A, B)\}_{(A,B) \in \mathsf{C}}$ is natural in C if and only if the swap isomorphisms $s_{S \oplus S}$ are *equalities* for all $S \in \mathsf{C}$.

Proof. Choose arbitrary $A, B, C, D \in S^{-1}S(\mathsf{C})$. If we denote by f an arbitrary morphism $(S, \alpha, \beta) : (A, B) \to (C, D)$, then $(t \oplus \mathrm{id})f$ is the morphism $(B \oplus A, A \oplus B) \to (D \oplus C, C \oplus D)$ consisting of the following two arrows.

$$B \oplus A \oplus S \oplus S \xrightarrow{\mathrm{id}_B \oplus s_{AS} \oplus \mathrm{id}_S} B \oplus S \oplus A \oplus S \xrightarrow{\quad \beta \oplus \alpha \quad} D \oplus C$$

$$A \oplus B \oplus S \oplus S \xrightarrow{\mathrm{id}_A \oplus s_{BS} \oplus \mathrm{id}_S} A \oplus S \oplus B \oplus S \xrightarrow{\quad \alpha \oplus \beta \quad} C \oplus D$$

[10] This signage is decided by the direction of edges in $S^{-1}S(\mathsf{C})$.

[11] If C is an exact category, you could also argue that the H-space is an H-group by proving that the map $K_0() \to \pi_0(BS^{-1}S)$ is an isomorphism and that the former is a group. I read this in **??**.

Naturality amounts to the commutativity of the following diagram, for all A, B, C, D, and f.

$$\begin{array}{ccc} (0,0) & \xrightarrow{\eta_{AB}} & (B \oplus A, A \oplus B) \\ {\scriptstyle 0(f)=0}\downarrow & & \downarrow{\scriptstyle (t \oplus \mathrm{id}) f} \\ (0,0) & \xrightarrow{\eta_{CD}} & (D \oplus C, C \oplus D) \end{array}$$

Writing out η_{CD} and the composite $(t \oplus \mathrm{id}) f \circ \eta_{AB}$, Thomason finds commutativity to be equivalent to that of

$$\begin{array}{ccc} BASS & \xrightarrow{s_{AS}} & BSAS \\ {\scriptstyle s_{AB}}\uparrow & & \uparrow{\scriptstyle s_{(AS)(BS)}} \\ ABSS & \xrightarrow{s_{BS}} & ASBS \end{array}$$

Note that this diagram is in C. Observe that up-right does not reverse the order of the two S's. However, right-up does (because $s_{(AS)(BS)}$ does). Hence, this diagram commutes if and only if the swap map $s_{S \oplus S}$ is a *strict* equality. As S was arbitrary, the result follows. $\square$

The swap isomorphisms are rarely equalities, and although it is possibly to "strictify" a symmetric monoidal category into a permutative one (in which associativity and unitality are strict), commutativity cannot be strictified except in trivial cases. Hence, the result outs (III) a tricky condition to satisfy, for the obvious natural transformation is *not* a natural transformation! A tragedy.

Let me quote some of Peter's commentary [May80]:

> Thomason [. . .] has given an amusing illustration of the sort of mistake that can arise from a too cavalier attitude towards this kind of categorical distinction when studying pairings of categories, and one of my concerns is to correct a similar mistake of my own.
>
> In [. . .], I developed a coherence theory of higher homotopies for ring spaces up to homotopy and for pairings of H-spaces. That theory is entirely correct. I also discussed the analogous categorical coherence, proving some results and asserting others. That theory too is entirely correct, my unproven assertions having been carefully proven by Laplaza [unpublished]. However, my translations from the categorical to the homotopical theories in [. . .], that of course being the part I thought to be obvious, are quite wrong.
>
> The moral is that to treat the transition from categorical coherence to homotopical coherence smoothly and rigorously, one should take advantage of the definitional framework established by the category theorists.

There's more to say. But I have been writing some time now, and have generated me thoughts I need to lay bare before the crystal ball (ask Charles about). Let me just say something about the first great victim of Thomason's observation.

Recall that if we put a semiring structure on a monoid M (i.e., a unital operation $\times$ distributing over $+$), for example that already present on the underlying monoid of any unital associative ring, then this straightforwardly extends to a ring structure on the group $K_0(M) = M^{\mathrm{gp}}$. In fact, the full K-theory $K_*(A)$ (of rings, schemes, whatever) has a ring structure. Back in the day, Quillen invented higher algebraic K-theory

Finish this. Talk about ring structure on $K(R)$, relation of Quillen

II.2 (2/7) Localizations II

Johnson and Jeremiah spoke for the (anti)telescope seminar today. Johnson wrapped up his previous talk (introducing monochromaticity in $\mathsf{Cat}_{\mathrm{perf}}$) with a proof that the category of *n-monochromatic categories* is ∞-semiadditive for all n. Jeremiah gave an overview of algebraic K-theory. We did all this with an eye toward [Ben+23] (wherein spawns the monochromatic language for categories). One takeaway is that I need to learn more about localizations (and the chromatic phenomena motivating our modern theory of localizations, c.f. my monologue last time).

References are:

- Moritz Groth's notes, again
- HTT and maybe HA, again
- Lawson's notes, again
- I also found Martin Gallauer's lecture notes here, although I have not looked at these yet.

Let's start with ordinary categories. We did some of this last time. There, we wanted to invert a class of morphisms $S \subseteq \mathrm{Mor}(\mathsf{C})$ in a universal way. We did not study existence, but we assumed existence and (i) showed uniqueness of a given object's S-localizations, (ii) exhibited a functorial choice of S-localizations $X \mapsto LX$, and (iii) thought about some basic examples. The resultant basic structure is an adjunction

$$\mathsf{C} \underset{\perp}{\overset{L^S}{\rightleftarrows}} L^S(\mathsf{C})$$

Which full subcategories arise in this manner? Define a *reflective subcategory* to be a full subcategory $\mathsf{D} \hookrightarrow \mathsf{C}$ whose inclusion admits a left adjoint. These are essentially our localizations. It will be a common theme that localizations can be understood as adjoint systems (existence, properties, etc.).

Proposition II.4. Let $(L \dashv i)$ denote an adjoint pair. TFAE.

(1) The right adjoint $i : \mathsf{D} \to \mathsf{C}$ is fully-faithful ($\Longleftrightarrow$ D is a reflective subcategory).

(2) The counit $Li \to \mathrm{id}_{\mathsf{D}}$ is a natural isomorphism.

(3) The left adjoint $L : \mathsf{C} \to \mathsf{D}$ is a localization at $S = L^{-1}(\text{isos})$.

Proposition II.5. Dually, define a *coreflective subcategory* to be a full subcategory whose inclusion admits a right adjoint. Let $(i \dashv R)$ denotes an adjoint pair. TFAE.

(1) The left adjoint $i : \mathsf{D} \to \mathsf{C}$ is fully faithful ($\Longleftrightarrow$ D is a coreflective subcategory).

(2) The unit $\mathrm{id}_{\mathsf{C}} \to iR$ is a natural isomorphism.

(3) The right adjoint $R : \mathsf{C} \to \mathsf{D}$ is a colocalization at $S = R^{-1}(\text{isos})$.

To my eye, the ordinary theory of localizations is a bit messy. Maybe I feel this way because I do not like how we handle comparisons/notions of equivalence. Maybe that's no discredit to the theory, just nLab's (lack of) presentation, and the fact that most literature I can find works in situations I am trying to not focus on. I'll just move on.

Toward real math, we want to think about localizations when C is *topologically enriched*. One asks whether this reduces to the localization theory of the homotopy category. The following example suggests this is an awkward approach to the theory.

Example II.1. Consider Top the topological category of spaces. Pass to its homotopy category and consider the map

$$i_n : S^n \hookrightarrow D^{n+1}.$$

We can ask about inverting i_n. This requires exhibiting i_n-localizations. Given a space X, form X' by gluing copies of D^{n+1} onto X to kill $\pi_n X$. Pullback along i_n induces an equivalence $[D^{n+1}, X'] \xrightarrow{\sim} [S^n, X']$ since both sides are trivial (in their path components) by construction, hence X' is i_n-local. Is the map $l : X \to X'$ an i_n-equivalence? This would require the pullback

$$l^* : [X', Y] \to [X, Y]$$

to be an isomorphism for every i_n-local space Y. This map is surjective: Y is i_n-local if and only if $\pi_n Y$ is zero, in which case an extension of any $f : X \to Y$ over any n-cell $\partial D^{n+1} \to X$ exists, exhibiting an inverse in $l^{-1}(f)$. The problem is uniqueness: if $\partial D^{n+1} \to X$ denotes a cell killed to form X', then given $f : X \to Y$, two extensions g, h of f to D^{n+1} determine a map $S^{n+1} \to Y$, and $g \simeq h$ if and only if this map is nullhomotopic. In this way, $\pi_{n+1} Y$ obstructs the injectivity of l^*. Since l^* is not injective, another candidate localization X'' may not be homotopy equivalent to X'.

Remark II.3. The example shows that we cannot solely invert $i_n : S^n \hookrightarrow D^{n+1}$ in $h\mathsf{Top}$, at least not nicely or obviously. The example also showed that inverting i_n requires at least inverting i_m for all $m \geq n$. One imagines that inverting an arbitrary map f comes with similar responsibilities as soon as f is not a homotopy equivalence (i.e., is not already inverted).

Thus, given a topological category C and a class $S \subseteq \mathrm{Mor}(\mathsf{C})$, rather than work with $h\mathsf{C}$, we define *S-local objects* and *S-equivalences* in C just as before except with "isomorphism" replaced with "weak equivalence." With these properties defined, we can define *S-localizations* just as before.

Proposition II.6. If $Y \in \mathsf{C}$ is S-local, then $hY \in h\mathsf{C}$ is hS-local. However, since hS is generally "smaller" than S, there are generally more hS-local objects. Thus, if f is an S-equivalence, it is not necessarily true that $[f]$ is an hS-equivalence.

We would lastly like a localization *functor*. Unfortunately, even if our topological category C has localizations, choosing them *naturally* is a coherence problem. When and how we can do this is nontrivial.

Spiel about small object argument, Lawson §5,6,7

Agony and confusion. How model category theory Bousfield did all this for spaces and spectra. His theory extends to model categories, and resultant localizations get a model structure too. Localization and completions of spectra are examples of Bousfield localizations. How/in what language (stable ∞-categories?) can we think about Bousfield localizations as "functors with certain kinds of adjoints," as we did using (co)reflective subcategories in the case of categorical localization? Maybe this is not the right way to think about it. Also, triangulated categories and Verdier localization—what's going on there? I think this is where the "thick subcategory" ideas originate. Maybe H. Krause's notes [Kra09] will clarify. How do things converge when you use stable ∞-categories?

Think about localizations of (based, unbased, G-) spaces. Fiberwise localization? Also, whatever Lawson talks about at the end of §7.

(Trigger warning: yapping.) We came to view ordinary localization as the search for left adjoints with fully faithful right adjoints. When considering a topological category C, we found it ineffective to restrict to the homotopy category $h\mathsf{C}$, so instead we carried out the story of ordinary localizations but with *weak equivalences*. Bousfield did this for spaces and spectra. But things are not so simple now—even if localizations exist, choosing them *functorially* is a nontrivial *coherence problem*. So that complicates things. More generally, this problem occurs for *model categories*. In nice situations, we can view this as a lifting problem, and somehow a functorial choice is possible when "the small object argument works." Localization becomes a process of changing model structures on a fixed category. This is somewhat different from how you might naturally present and study localizations in ordinary category theory via universal properties. I think you can frame this as an adjoint systems phenomenon *in the language of Quillen adjunctions*, but extracting a plain categorical statement from this is hard? This presents other issues too—e.g., the extension of monoidal structures is very nontrivial.

In contrast, it is quite easy to express and organize ∞-categorical localizations via universal properties and adjoint systems. This is maybe because the semantics of ∞-categories, functors, etc. are designed to "see" and naturally handle the associated lifting problems. We can e.g. stay faithful to the story of "reflective subcategories."

Definition II.2. Let C, D denote ∞-categories. A functor $\mathsf{C} \to \mathsf{D}$ is called a *localization* if it admits a fully faithful right adjoint. We may call D a *localization of* C and we may call the resultant functor $\mathsf{C} \to \mathsf{C}$ a *localization functor*.

Proposition II.7 (Basic form of localizations, [Lur08, 5.2.7.4])**.** Let C be an ∞-category and $L : \mathsf{C} \to \mathsf{C}$ a functor with essential image $L\mathsf{C}$. TFAE.

(1) There exists a localization $f : \mathsf{C} \to \mathsf{D}$ with fully faithful right adjoint g and an equivalence $g \circ f \simeq L$.

(2) As a functor $L : \mathsf{C} \to L\mathsf{C}$, L is left adjoint to the inclusion $L\mathsf{C} \hookrightarrow \mathsf{C}$.

(3) There exists a natural transformation $\alpha : \mathsf{C} \times \Delta^1 \to \mathsf{C}$ from id_{C} to L such that the morphisms $\alpha(LC), L(\alpha C) : LC \to LLC$ are equivalences.

Proposition II.8 ([Lur08, 5.2.7.8])**.** Consider $\mathsf{C}_0 \hookrightarrow \mathsf{C}$ an ∞-category and a full subcategory. TFAE.

(1) The inclusion $\mathsf{C}_0 \hookrightarrow \mathsf{C}$ admits a left adjoint.

(2) Each object $C \in \mathsf{C}$ admits a morphism $f : C \to D$ which *exhibits D as a C_0-localization of C*. By this, we mean that $D \in \mathsf{C}_0$ and for every $E \in \mathsf{C}_0$, the map $f^* : \mathrm{Map}_{\mathsf{C}_0}(D, E) \to \mathrm{Map}_{\mathsf{C}}(C, E)$ is an equivalence in the homotopy category.

Remark II.4. Note that $f : C \to D$ exhibits D as a C_0-localization if and only if $f \in \mathsf{C}_{C/} \times_{\mathsf{C}} \mathsf{C}_0$ is an initial object. This implies e.g. uniqueness and invariance under equivalence of categories.

If a full subcategory $\mathsf{C}_0 \subseteq \mathsf{C}$ satisfies either of the equivalent conditions of the above proposition, we call C_0 a *reflective subcategory*. It is clear that reflective subcategories are an equivalent formulation of ∞-categorical localizations.

Proposition II.9 ([Lur08, 5.2.7.12])**.** Let $L : \mathsf{C} \to \mathsf{C}$ be a localization with essential image $L\mathsf{C}$. Then for any $f : \mathsf{C} \to \mathsf{D}$, composition induces an equivalence

Wait, what is this proposition saying?

Recall that an ∞-category is called *presentable* if it is accessible and admits small colimits. Presentable ∞-categories have a nice theory of adjunctions. For this and other reasons, they have a nice theory of localizations. I'll quote [Lur08, §5.5]:

> In view of Theorem 5.5.1.1, the theory of localizations plays a central role in the study of presentable ∞-categories. In §5.5.4, we will show that the collection of all (accessible) localizations of a presentable ∞-category C can be parametrized in a very simple way. Moreover, there is a good supply of localizations of C: given any (small) collection of morphisms S of C, one can construct a corresponding localization functor
> $$C \xrightarrow{L} S^{-1}\mathsf{C} \subseteq \mathsf{C}$$
> where $S^{-1}\mathsf{C}$ is a the full subcategory of C spanned by the S-local objects. These ideas are due to Bousfield, who works in the setting of model categories; we will give an exposition here in the language of ∞-categories. In §5.5.5, we will employ the same techniques to produce examples of factorization systems on the ∞-category C.

I need more time to read §5 of HTT, but here's an important classification result. We say that a localization L is *accessible* if its fully faithful right adjoint is *accessible*, i.e. if it preserves all κ-filtered colimits for some regular cardinal κ.

Theorem II.1. An ∞-category C is presentable if and only if it is an accessible localization of $\mathrm{PShv}(\mathsf{D})$ for some small ∞-category D.

Remark II.5. Recall that presentable ∞-categories enjoy a (left) adjoint functor theorem: if C, D are presentable, then a functor $F : \mathsf{C} \to \mathsf{D}$ is a left adjoint if and only if it preserves small colimits. However, the converse needs an additional hypothesis: F is a right adjoint if and only if it preserves small limits *and is accessible*. In particular, if $L\mathsf{C} \to \mathsf{D}$ is a localization, then its right adjoint is automatically accessible. This explains the appearance of accessibility in the classification result.

Given a localization $F : \mathsf{C} \to \mathsf{D}$, we denote by $L : \mathsf{C} \to \mathsf{C}$ the composition, by which we can study this localization. As in ordinary category theory, the "local" objects play an essential role. Denote by S_L the collection of morphisms inverted by L. Say an object $c \in \mathsf{C}$ is *S_L-local* if for all $f \in S_L$, the induced $f^* : \mathrm{Map}(B, c) \to \mathrm{Map}(A, c)$ is weak equivalence.

Proposition II.10 ([Lur08, 5.5.4.2])**.** Given C and localization functor $L : \mathsf{C} \to \mathsf{C}$, the essential image $L\mathsf{C}$ is spanned by the S_L-local objects.

As a localization, L is left adjoint to the inclusion $L\mathsf{C} \hookrightarrow \mathsf{C}$. The proposition identifies $L\mathsf{C} = S_L$, thus L is determined up to equivalence by S_L. Hence, localizations correspond to certain classes of morphisms. These classes are not arbitrary, but rather possess nice closure properties. An arbitrary class of morphisms having such closure properties (e.g., closure under colimits, pushdowns, 2-out-of-3) is called *strongly saturated* [Lur08, 5.5.4.5]. If C admits small colimits, then $\mathrm{Mor}(\mathsf{C})$ is strongly saturated, and moreover the intersection of srongly saturated classes is strongly saturated, whence any collection S_0 has a minimal containing strongly saturated class $\overline{S_0}$. We may say $\overline{S_0}$ is the strong class of morphisms *generated by S_0* or call it the *strong saturation* of S_0. If S_0 is small, we say $\overline{S_0}$ is *of small generation*.

This language established, we can systematically describe the reverse process: specifying a set of morphisms and exhibiting a localization.

Proposition II.11 ([Lur08, 5.5.4.15])**.** Let C be a presentable ∞-category and S a small class of morphisms. Let $L^S\mathsf{C}$ denote the full subcategory of S-local objects. Then:

(1) $L^S\mathsf{C}$ is presentable.
(2) The inclusion $L^S\mathsf{C} \hookrightarrow \mathsf{C}$ has a left adjoint L. (Which, by the right adjoint functor theorem for presentable categories, implies that the inclusion is accessible.)
(3) Each $c \in \mathsf{C}$ admits an S-equivalence $c \to c'$ to some $c' \in L^S\mathsf{C}$.
(4) TFAE.
    (a) f is an S-equivalence.
    (b) f is in $\overline{S}$.
    (c) Lf is an equivalence.

We knew that localizations $L : \mathsf{C} \to \mathsf{C}$ of an arbitrary ∞-category C correspond to certain classes of morphisms (which occur as $L^{-1}(\text{isos})$). The proposition says that if C is presentable, then this correspondence can be made simple and precise. In one direction, given *any* set of morphisms S, there exists an associated accessible localization $\mathsf{C} \xrightarrow{L} L^S\mathsf{C} \hookrightarrow \mathsf{C}$ inverting $\overline{S}$, and all localizations arise this way. In the other direction, given a localization L, we extract $S = L^{-1}(\text{isos})$, which is a strongly saturated set of small generation. This is not a bijective process, but noting that $\overline{S} = \overline{T} \Longrightarrow L^S\mathsf{C} = \mathrm{L}^T\mathsf{C}$, we get the following.

Proposition II.12. The accessible localizations of presentable ∞-category C correspond with its strongly saturated classes of small generation.

Example II.2. Recall that if C is pointed with finite limits and finite colimits, we can define its category of *prespectrum objects* and *spectrum objects*. These are modeled concretely and intuitively, c.f. Section I.4. Using this concrete model, it is easy to take a prespectrum X and freely construct a "spectrum below level n" L_nX. We would like to consider $\mathrm{colim}_n L_n X$ as the *spectrafication* of X. With some mild hypotheses on C (c.f. Proposition I.7), this works: $L := \mathrm{colim}_n L_n$ is a localization $\mathsf{PSp}(\mathsf{C}) \to \mathsf{PSp}(\mathsf{C})$ with essential image $\mathsf{Sp}(\mathsf{C})$. In particular, taking $\mathsf{C} = \mathsf{Spaces}_*$, we find Sp as a localization of PSp. Since Spaces_* is presentable, this exhibits Sp as an (accessible) localization of a presentable category, whence Sp is presentable.

What's the right way to see that Sp is pre-

II.3 (2/14) Categorifying Mahler's theorem

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Some references for today are:

- Lior Yanovski's thesis on homotopy cardinality and Euler characteristics, which motivated this post.
- Noam Elkie's short note about his generating function proof of Mahler's theorem.
- Bibby-Badish's article using generating functions to study the homology of orbit configuration spaces.
- Mahler's 1958 paper [Mah58] giving critertia for continuously extending p-adic functions.

Given a sequence of numbers $A = \{a_i\}$, we define its *binomial transform* as the sequence TA given by

$$(TA)_n := \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k.$$

This defines an involution T of $\mathrm{Fun}(\mathbb{N}, M)$ for any $\mathbb{Z}$-module M.[12] The binomial transform is closely related to the *forward difference operator* Δ given by $(\Delta A)_n = a_{n+1} - a_n$.

Example II.3. Let me write out the first few terms of some iterates of $\Delta^k A$.

$$\begin{aligned} \Delta A &= a_1 - a_0, \quad a_1 - a_2, \quad \dots \\ \Delta^2 A &= a_0 - 2a_1 + a_2, \quad a_3 - 2a_2 + a_1, \quad \dots \\ \Delta^3 A &= a_3 - 3a_2 + 3a_1 - a_0, \quad \dots \end{aligned}$$

The relation between Δ and T is clear: $(\Delta^n A)_0 = (TA)_n$.

With that notation established, a classical problem: how to estimate a function? If $f : \mathbb{R} \to \mathbb{R}$ is smooth, we may consider its Taylor power series $\sum f^{(k)}(x)x^k/k!$ about zero. This power series does not always converge, nor does it always converge to f. We may also consider the *Newton series* of f, defined as a discrete Taylor series:

$$Nf(x) := \sum_{k=0}^{\infty} \frac{(\Delta^k f)(0)}{k!} x(x-1)(x-2)\dots(x-k+1).$$

This does not always converge either. What's interesting is that p-adically, *the obstruction to coincidence of Nf with f is continuity.* Compare to the real case, where not even smoothness guarantees convergence, let alone coincidence.

Proposition II.13 (Mahler's theorem)**.** Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a p-adic function. TFAE.

(1) f is continuous.

(2) The Newton series for f converges to f everywhere, i.e. $f(x) = \sum_{k=0}^{\infty} \binom{x}{k}(\Delta^k f)(0)$.

(3) The sequence of coefficients $(Tf)_k = (\Delta^k f)(0)$ converges to $0 \in \mathbb{Q}_p$.

[12] $\mathrm{Fun}(\mathbb{N}, M)$ is an abelian group under pointwise addition and T is a group homomorphism. I have several questions...

Given a sequence $A = \{a_i \in \mathbb{Q}_p\}_{i=0}^{\infty}$, we may consider its binomial transformation $B = TA$, and if $b_k \to 0$ as $k \to \infty$ then the p-adic power series $A(x) := \sum \binom{x}{k} b_k$ is a uniform limit of polynomials. It therefore converges to a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$ satisfying $A(n) = a_n$. This describes a process for extending a "nice" function $\mathbb{N} \to \mathbb{Q}_p$ to a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$. This is how you go from (3) to (1). Here, "nice" is the (above) stated convergence condition on the binomial transform BA.

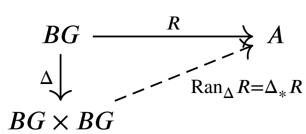
Finish this

II.4 (2/21) The pushdiagonal of a ring with G-action

Let C be a nice ∞-category (I will be more specific later). I want to think about functors $BG \to A := \mathsf{CAlg}(\mathsf{C})$. For now, just take $\mathsf{C} = \mathsf{Mod}_k$ for some field k, so that such a functor R is precisely a commutative k-algebra with G-action (by k-linear automorphisms). We can ask about the pushforward

$$BG \times BG \xrightarrow{\Delta_* R} A$$

where $\Delta : BG \to BG \times BG$ is the diagonal functor. The functor Δ_* is right adjoint to the pullback Δ^* and hence is characterized as the right Kan extension of R along Δ, as in the following diagram.



The right Kan extension exists because A has limits, and it is computed by fiberwise limits, i.e.

$$[\Delta_* R](*) = \lim_{\Delta_{/*}} R\pi.$$

Here, $\Delta_{/*}$ is the slice of Δ over $* \in BG \times BG$ and $\pi : \Delta_{/*} \to BG$ is the projection. This is the general formula for a right Kan extension. Let's acknowledge BG's elementary structure and simplify the formula: the slice category $\Delta_{/*}$ has objects $(g_1, g_2) \in G \times G$ and has a morphism $h : (g_1, g_2) \to (g_1', g_2')$ for each h such that $g_1 = g_1' h$ and $g_2 = g_2' h$. Each object $(g_1, g_2) \in \Delta_{/*}$ is uniquely isomorphic via g_1 to $(e, g_2 g_1^{-1})$, hence $\Delta_{/*}$ is canonically equivalent to a discrete subcategory isomorphic to G (embedded on objects by $g \mapsto (e, g)$). We get a much more familiar expression for the underlying object of $\Delta_* R : BG \times BG \to A$:

$$[\Delta_*(R)](*) = \lim_{G} R\pi = \prod_{G} R(*).$$

Just as the underlying object of $R : BG \to A$ carries a G-action, that of $\Delta_* R$ carries a $G \times G$ action. That action is given by $(x, y) r_g = x r_{gy^{-1}}$.

II.5 (2/22) Linear Galois theory I

> An absolutely natural impulse in virtually all of algebra is to do for commutative rings what has already been done for fields.
>
> Daniel Zelinsky

Some references for today are

(1) Rognes' seminal *Galois extensions of structured ring spectra* [Rog05],

(2) Gow-Quinlan's article "Galois theory and linear algebra" [GQ09],

(3) Michael Francis's notes on "Linear Galois theory" found on his site here,

(4) Sharon Zhou's great Chicago REU paper found here,

(5) Farb-Dennis *Noncommutative Algebra* [FD12], and

(6) Dress's article "One more shortcut to Galois theory" [Dre95].

Recall Galois theory. A field extension L/K is called a *Galois extension* if it is algebraic, normal, and seperable. There are a lot of ways to characterize and study Galois extensions, especially finite ones. For instance, finite Galois extensions of K are precisely the splitting fields of seperable polynomials $f \in K[x]$. Alternatively, the *fundamental theorem of Galois theory* says that a finite extension L/K is Galois if and only if the following correspondence (I won't write it out) is bijective.

$$\{\text{intermediate field extensions } L/E/K\} \longleftrightarrow \{\text{subgroups of } \mathrm{Aut}(L/K)\}$$

If L/K is not finite, we can still define the maps both ways, but the analogous statement is more nuanced. The main detail is that $\mathrm{Aut}(L/K)$ is naturally profinite, hence carries a *profinite topology*, and taking Galois groups of intermediary extensions must produce *closed* subgroups in this topology. Luckily, we can refine the fundamental theorem: a (possibly infinite) field extension L/K is Galois if and only if the following restricted correspondence is bijective.

$$\{\text{intermediate field extensions } L/E/K\} \longleftrightarrow \{\textit{closed} \text{ subgroups of } \mathrm{Aut}(L/K)\}$$

Another formulation of finite extensions emphasizes the Galois group. Given a finite, *faithful* group action $G \to \mathrm{Aut}(L)$, the extension L/L^G is Galois, in fact finite with Galois group $G = \mathrm{Aut}(L/L^G)$! That's a bit funny—faithfulness is equivalent to the *injectivity* of $G \to \mathrm{Aut}(L)$, but it is not in my nature to suspect this restricts to an *equivalence* $G \cong \mathrm{Aut}(L/L^G)$ (whence L/L^G is Galois, in fact finite). This is proven in [Sta24, Lemma 09I3]. Conversely, given an extension L/K, its Galois group acts faithfully on L. Therefore, for a fixed L, we have

$$\text{finite Galois extensions } L/K \quad \cong \quad \text{finite, faithful group actions } G \hookrightarrow \mathrm{Aut}(L).$$

I think the picture holds for arbitrary Galois extensions if you consider faithful actions by profinite groups.

(Story about Brauer groups, extensions, and Galois extensions?)

We have many approaches to the Galois theory of fields. As to that of *commutative rings*, we are fat with food for thought. What is the correct generalization of Galois theory to rings? Moreover, what would conveniently extend to ring *spectra*? For this, we will bring to the fore the *linear algebra* secretly powering the classical Galois theory of fields. You have already seen footprints of linear algebra in Galois theory, e.g. in dimension counting arguments. Following Francis, we will set up Galois theory using only the linear independence of characters, Dedekind's theorem (which is an immediately consequence of the independence of characters), and Artin's lemma.

Lemma II.1 (Linear independence of characters)**.** Let G be a group and L a field. Then the set of characters $\mathrm{Hom}_{\mathsf{Gp}}(G, L^\times)$ is a linearly independent subset of the L-vector space $\mathrm{Fun}_{\mathsf{Set}}(G, L)$.

Every nontrivial element of a field is invertible, i.e. $K^\times = K - \{0\}$, and ring maps must preserve zero, therefore restriction $r^* : \mathrm{Hom}(K, L) \to \mathrm{Hom}_{\mathsf{Gp}}(K^\times, L)$ loses no information, i.e. r^* is injective. Moreover, these are both L-vector spaces and r^* is linear, whence the linear independence of characters extends to $\mathrm{Hom}(K, L)$.

Corollary II.1 (Dedekind's lemma)**.** Let $\{\sigma_i : K \to L\}$ denote a finite set of distinct field homomorphisms. Then $\{\sigma_i\}$ is L-linearly independent. In other words, $\mathrm{Hom}(K, L)$ is L-linearly independent as a subset of $\mathrm{Fun}_{\mathsf{Set}}(K, L)$. In particular, $\mathrm{Aut}(K)$ is K-linearly independent.

Now consider a subextension $F \hookrightarrow K \hookrightarrow L$. Field morphisms $K \to L$ fixing F are F-linear, that is to say $\mathrm{Hom}(K, L; /F) \subseteq \mathsf{Vect}_F(K, L)$. The latter space's L-dimension is $[K : F]$. Dedekind's lemma says that $\mathrm{Hom}(K, L; /F)$ is L-linearly independent, hence this set is no bigger than that dimension.

Proposition II.14. Let $F \hookrightarrow K \hookrightarrow L$ denote a subextension of fields. Then the number of field maps $K \to L$ fixing F is bounded by the L-dimension of $\mathsf{Vect}_F(K, L)$. In particular, taking $K = L$, we find that the number of automorphisms of K/F is so bounded. That is to say, in equations:

$$|\mathrm{Hom}(K, L; /F)| \leq [K : F] \quad \text{and}$$
$$|\mathrm{Aut}(L/F)| \leq [L : F].$$

One characterization of finite Galois extensions recognizes their achievement of equality in Proposition II.14. For the purpose of "inventing Galois theory linearly," we can make this our definition: say L/F is a *finite Galois extension* if $|\mathrm{Aut}(L/F)| = [L : F]$, which (by Dedekind's lemma) is equivalent to saying that $\mathrm{Aut}(L/F)$ constitutes a basis for the L-vector space $\mathsf{Vect}_F(L, L)$.

Thus, the basic language takes only linear algebra. What about the basic structure? Let's see what the orbit-stabilizer theorem has to say. Let me state the version I will use: if G acts on X and $S \subseteq X$ is a subset, then $|G| = |\mathrm{Stab}(S)| \cdot |\mathrm{Orb}(S)|$ where $\mathrm{Orb}(S)|$ denotes the embeddings $S \hookrightarrow X$ of the form $s \mapsto gs$ for some fixed g. We want to take this with $G = \mathrm{Aut}(L/F)$ acting on L with distinguished subset K.

Proposition II.15 (Orbit-stabilizer implies subextension structure)**.** Let $F \hookrightarrow K \hookrightarrow L$ denote a subextension of fields. Then we have

$$|\mathrm{Aut}(L/F)| \leq [L : K][K : F].$$

Furthermore, if L/F is Galois, then so is L/K, and the above inequality is an equality.

Proof. Take $G = \mathrm{Aut}(L/F)$, $X = L$, and $S = K$. Then the orbit-stabilizer theorem says that

$$|\mathrm{Aut}(L/F)| = |\underbrace{\mathrm{Stab}(K)}_{=\mathrm{Aut}(L/K)}| \cdot |\underbrace{\mathrm{Orb}(K)}_{=\{\phi|_K : \phi \in \mathrm{Aut}(L/F)\} \subseteq \mathrm{Hom}(K,L;/F)}|.$$

Noting what I've indicated with underbraces, we find $|G| \leq [L : K][K : F]$ by Proposition II.14. This proves the first part of the proposition. Now if L/F is Galois, then by definition $|\mathrm{Aut}(L/F)| = [L : F]$. Since $[L : F] = [L : K][K : F]$, the desired equalities follow. □

Corollary II.2. Given a subextension $L/K/F$, we know that $\mathrm{Orb}_{\mathrm{Aut}(L/F)}(K)$ sits within $\mathrm{Hom}(K, L; F)$. (Recall $\mathrm{Hom}(K, L; F)$ is L-linearly independent by Dedekind's lemma, thus $|\mathrm{Hom}(K, L; F)| \leq [K : F]$.) Our proof argues that if L/F is Galois, then among other things we have $|\mathrm{Orb}(K)| = [K : F]$, necessitating $\mathrm{Orb}(K) = \mathrm{Hom}(K, L; F)$.

In other words, we showed that i L/F is Galois, then every field morphism $K \to L$ fixing F extends to an automorphism of L, i.e. restriction $\mathrm{Aut}(L/F) \to \mathrm{Hom}(K, L; F)$ is surjective.

Remark II.6. This setup explains why composing Galois extensions is tricky. Suppose L/K and K/F are finite Galois extensions. Orbit-stabilizer says that $|\mathrm{Aut}(L, F)| = |\mathrm{Aut}(L/K)||\mathrm{Orb}(K)|$ which equals $[L : K] \cdot |\mathrm{Orb}(K)|$ since L/K is Galois. Thus L/F is Galois if and only if $|\mathrm{Orb}(K)| = [K : F]$.

By virtue of subextensions, we have an inclusion $\mathrm{Orb}(K) \hookrightarrow \mathrm{Hom}(K, L; F)$, which establishes $|\mathrm{Orb}(K)| \leq [K : F]$. Then L/F is Galois if and only if this is surjective, i.e. iff every $K \to L$ fixing F extends to an automorphism $L \to L$. This is a nontrivial extension problem.

This so far has not acknowledged our assumption that K/F is Galois. That implies $[K : F] = |\mathrm{Aut}(K/F)|$, so K/F being Galois means L/F is Galois iff $|\mathrm{Orb}(K)| = |\mathrm{Aut}(K/F)|$. Note that $\mathrm{Aut}(K/F)$ is also a subset of $\mathrm{Hom}(K, L; /F)$ via postcomposing $K \hookrightarrow L$, so L/F is Galois iff $\mathrm{Orb}(K) \hookrightarrow \mathrm{Hom}(K, L; F)$ has image $\mathrm{Aut}(K/F)$. For this to happen, everything must coincide anyway, i.e. $\mathrm{Orb}(K) = \mathrm{Aut}(K/F) = \mathrm{Hom}(K, L; F)$, so we've not managed to make our problem any easier by assuming K/F is Galois.

Theorem II.2 (Fundamental theorem of Galois theory)**.** Let L/F be a finite Galois extension. Then (blah blah blah)

Proof. Prove with the above + Artin's lemma. Not hard. Essential detail is that for $L/K/F$, have equality $K = L^{\mathrm{Aut}(L/K)}$, and that for finite $H \leq \mathrm{Aut}(L)$, have $H = \mathrm{Aut}(E/E^H)$. □

Maybe finish writing the above. Ring-theoretic correspondence?

II.6 (2/26) Linear Galois theory II

Last time, we worked up to the fundamental theorem of Galois theory using basic linear algebra and some group theory, following Michael (c.f. the references from last time). This was both a refresher of my Galois theory and an introduction to the linear algebra in Galois theory. The point is to work up to the Galois theory of rings and ring spectra, which "emphasizes (takes as definition) the linear algebraic point of view." (Although, as I continue to learn things, I'm less convinced this is the correct slogan.) But what exactly does that mean? Unfortunately, the picture is not so simple as to somehow seamlessly extend the previous post's approach. I knew going in that $\mathrm{End}_K(L)$ played some crucial role, and I wanted the previous post to elucidate this. We *did* get some $\mathrm{End}_K(L)$ action last time (c.f. the role of $\mathsf{Vect}_K(L, L) = \mathrm{End}_K(L)$), but maybe not enough. (That is not to say the last post was not productive, just maybe did not progress my point.) I've since found some more good references:

- Greither's book *Cyclic Galois extensions of commutative rings* [Gre06] which I accessed here,
- Keith Conrad's notes "Linear independence of characters" found here, and
- Keith Conrad's notes "Galois descent" found here, and
- Baez's nLab post about group cohomology, homotopy fixed points, and Galois descent, and
- This MO question and the links therein.

Let's keep working on this. That is, let me keep trying to *find* $\mathrm{End}_K(L)$ and ultimately the definition of *Galois extensions of rings*, rather than serve it up instantly and mysteriously.

Let me return to my earlier comment that one can take an approach to Galois theory "emphasizing the Galois group." That is, we can characterized finite Galois extensions as faithful finite group actions $G \to \mathrm{Aut}(L)$. Such a thing encodes an extension L/L^G with the (funny) property that $\mathrm{Aut}(L/L^G) = G$. (That property is precisely the magic in classical the Galois correspondence.)

This approach is a little too slick. Recall that Galois extensions are characterized as the extensions which are algebraic, normal, and separable. These are essential properties—both for how they interact to effect Galois extensions and for their independent importance—but it is not obvious how these properties manifest in a faithful action $G \to \mathrm{Aut}(L)$, let alone how to isolate and study them.

Proposition II.16.

II.7 (2/28) Descent I

I have been trying to understand Galois extensions of ring spectra. The definition uses an interpretation of Galois extensions that was, upon my initial reading, unfamiliar to me. Somehow, my impression is that this interpretation is closely related to "descent philosophy." E.g., my hunch is that a baby case of descent explains the funny fact that if a finite group G acts by automorphisms on L, then L/L^G is G-Galois. Unclear to me is the precise relation between descent properties and Galois extensions—an example question is, *are the Galois extensions (using this definition) somehow precisely those satisfying (some form of) descent?* Or is the relation just that *when we take this as the definition of Galois extensions, (Galois) descent is more naturally "available"?*

Descent is something I have wanted to understand for a while but which I have avoided. I think I will try to get into it now. But even though Galois theory most recently brought me to think about descent, I will approach it from a more elementary point than is necessary for my earlier purposes, at least to start. I do this in part because I think I will learn it better, more generally, and hopefully in a way that sets up for categorical machinery.

Here are some references I dug up this morning, certainly enough to keep me busy for a while:

- Whatever is in HTT [Lur08], haven't checked but I'm sure it's in there,
- Charles' 2019 Leeds lectures on higher topos theory, the notes for which I find on his website,
- Hess's "A general framework for homotopic descent and codescent" [Hes10],
- Caenepeel's "Galois corings from the descent theory point of view" [Cae03],
- Hohl's "An introduction to field extensions and Galois descent for sheaves of vector spaces" [Hoh23],
- SheafifiedSarah's blog post about descent in graph theory here,
- Keith Conrad's notes "Galois descent" found here,
- Baez's nLab post about group cohomology, homotopy fixed points, and Galois descent, and all the links therein; and
- Vistoli's notes [Vis07].

III March

III.1 (3/1) The Eckmann-Hilton argument and some consequences

I am on a train to Chicago right now. I want to think about the Eckmann-Hilton argument to pass the time. Here is a question: *why is $\pi_2(X, x_0)$ abelian?* Let's start by understanding $\pi_2(X, x_0)$ as the pointed set of homotopy classes of maps $[0,1]^2 \to X$ satisfying $\partial[0,1]^2 \mapsto x_0$. Then you define (say) *horizontal concatenation* $+_h$ of maps, and the standard argument for commutativity proceeds pictorially: you realize the two-dimensionality of $[0,1]^2$ gives you enough space to spin $f +_h g$ into $g +_h f$. Thus, horizontal operation $+_h$ defined on $\pi_2(X, x_0)$ is commutative. Alternatively, you could have considered *vertical composition* $+_v$ and argued it is commutative.

It is clear that these are both associative operations with inverses, and moreover *the same operation*. But suppose this was not clear, and rather that we just had before us the two unital, associative operations $\times_h$ and $\times_v$. Recall our pictorial argument that $\times_h$ is commutative, and in your minds eye see that as we spin $f \times_h g$ into $g \times_h f$, we basically exhibit $f \times_h g = f \times_v g$. Likewise, as we spin $f \times_v g$ into $g \times_v f$ to prove commutativity, we end up exhibiting $f \times_v g = f \times_h g$. It seems that because $\times_h$ and $\times_v$ are "related by spinning," we can deduce their coincidence and commutativity. The *Eckmann-Hilton argument* says that this is true in a precise, algebraic sense.

Proposition III.1 (Eckmann-Hilton argument)**.** Let M be a set and let $\bullet$ and $\circ$ denote two monoidal operations on M. Then the following are equivalent.

(I) The operations $\bullet$ and $\circ$ are equal and commutative.

(II) The operations $\bullet$ and $\circ$ commute, in the sense that $(a \bullet b)\circ(c \bullet d) = (a\circ b) \bullet (c\circ d)$.

(III) The operation $\bullet : (M, \circ)^2 \to (M, \circ)$ is a morphism of monoids.

(IV) The operation $\bullet$ makes $(M, \circ)$ a monoid object in the monoidal category $(\mathsf{Mon}(\mathsf{Set}, \times), \times)$.

It is obvious that (I) $\implies$ (II), (III), and (IV). It is also obvious that (II) $\iff$ (III). It is also obvious that (III) $\iff$ (IV) once you unwind definitions. The interesting statement is (II) $\implies$ (I). The slogan is that if two monoidal operations commute, then they are equal and commutative. The proof is easy and straightforward. In categorical language, the Eckmann-Hilton argument can be stated as follows.

Proposition III.2. The forgetful functor $\mathsf{Mon}(\mathsf{Mon}) \to \mathsf{Mon}$ is fully faithful with essential image CMon.

This generalizes to arbitrary categories. The essential property was that the cartesian product is symmetric monoidal—a monoidal structure is needed to form Mon, and its symmetry is needed to induce a monoidal structure on Mon and thus to form $\mathsf{Mon}(\mathsf{Mon})$. (This turns out to be symmetric too.) That in mind, we get the following.

Proposition III.3 (Categorical Eckmann-Hilton)**.** If C is symmetric monoidal, then the forgetful functor $\mathsf{Mon}(\mathsf{Mon}(\mathsf{C})) \to \mathsf{Mon}(\mathsf{C})$ is fully faithful with essential image $\mathsf{CMon}(\mathsf{C})$.

Remark III.1. Note that $\mathsf{Mon}(\mathsf{Mon}) = \mathsf{CMon}$ actually has two monoidal structures. One is the cartesian product, the other is the *tensor product of commutative monoids*. By Eckmann-Hilton, nothing happens if we pass from CMon to its $\times$-monoids. However, $\otimes$-monoids are interesting. They are *rigs*. By the same procedure, we can talk about *rig objects* in any symmetric monoidal category.[13]

Now we can derive some consequences. One nice property of monoids is that the collection of functions to a monoid form a monoid under pointwise operation.

Proposition III.4. Suppose that C admits finite (co)products, is cartesian closed, and that M is a commutative monoid in C. Then for every object S, the mapping object M^S is a commutative monoid in C.

[13] Am I saying everything correctly?

Proof. This is actually easy to prove without the assumption that C has coproducts, but I do not know how to do this *using the Eckmann-Hilton argument*. But if you assume C has finite coproducts, you can show that $X \to X \times X \to X \coprod X \to M$ and $X \to M \times M \to M$ give rise to two monoidal, commuting operations on M^S. Then Eckmann-Hilton implies they are equal and commutative. C.f. my MSE question. □

You can also ask about endomorphism objects. Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category and consider End(1). This is a monoid under composition. But given $f, g \in \mathrm{End}(1)$, the powers that present another endomorphism

$$1 \xrightarrow{\sim} 1 \otimes 1 \xrightarrow{f \otimes g} 1 \otimes 1 \xrightarrow{\sim} 1.$$

One can show that this defines a monoidal operation on End(1) commuting with composition, hence Eckmann-Hilton implies these are the same operation and End(1) is commutative.

Proposition III.5. If $(C, \otimes, 1)$ is a monoidal category, then End(1) is a commutative monoid.

Proof. Write the (left) unitors as $\lambda_X : 1 \otimes X \xrightarrow{\sim} X$. As above, given $f, g \in \mathrm{End}(1)$, define $f * g := \lambda_1^{-1} \circ (f \otimes g) \circ \lambda_1$. This defines a monoidal operation on End(1). Now given $f, g, s, t \in \mathrm{End}(1)$, the functoriality of $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ requires preservation of compositions, which exactly says that

$$(f \otimes g) \circ (s \otimes t) = (f \circ g) \otimes (s \circ t).$$

□

IV April

IV.1 (4/3) THH, free loop spaces, and cochains do not commute with shearing

What a sorry title. I went to Mainz for a week to learn about unstable motivic stuff, then I went to Harvard's open house, which was a lot of fun. Then I had to prepare a talk for our "telescope seminar" at UIUC, for which I spoke about [Bur+23, §3]. That was hard and fun. I gave the talk today and finally have some free time. Stockholm syndrome drives me to write about something I learned while preparing my talk.

Let X be a nice space (I'm not sure what exactly to demand) and consider its free loop space $LX = \mathrm{Map}(S^1, X)$. A choice of basepoint $x_0 \in X$ determines a fiber sequence

$$\Omega X \to LX \to X.$$

Sometimes this fiber sequences splits, so that we get a weak equivalence $LX \simeq X \times \Omega X$. If we ask that $X = BG$, then this splitting occurs precisely when G is abelian. In that case, I think the basic idea is to (1) use the group structure to form a map $X \times \Omega X \to LX$ which informally "takes (g, f) and conjugates the based loop f with the path from x_0 to g" then (2) use commutativity to show this is an equivalence. Generalizing the ideas here to identify more cases where LX splits seems interesting; the general problem of determining when LX splits seems interesting and hopeless.

We will think in the general setting of an ∞-category C with finite limits. Denote by $*$ an initial object (an empty limit). Let's make some basic definitions and descriptions.

Definition IV.1. For an object $X \in \mathsf{C}$, we define the *free loop space LX* on X as a pullback

$$\begin{array}{ccc} LX & \longrightarrow & X \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{\ \Delta\ } & X \times_X X \end{array}$$

Proposition IV.1. Given a pointed object $(X, x) \in \mathsf{C}_*$, its based loop space admits two characterizations: it is the fiber $\{(x, x)\} \times_{X \times X} X$ and the fiber $\{x\} \times_X LX$.

Proposition IV.2. The free loop space LX is the limit of the constant S^1-shaped diagram

$$LX = \lim(S^1 \to \{X\} \to \mathsf{C}).$$

Proof. The point is that in the ∞-category of spaces, the circle S^1 is the suspension of S^0, i.e. $S^1 = * \sqcup_{* \sqcup *} *$. Then we can write $\lim(S^1 \to \{X\} \to \mathsf{C}) = X \times_{X \times X} X = LX$. □

Now we can talk about a phenomenon I will call *shearing* and its dual *coshearing*. A cartesian monoid $G \in \mathsf{Mon}(\mathsf{C})$ may be *grouplike*, which amounts to the maps $(a, b) \mapsto (a, ab)$ and $(a, b) \mapsto (ab, b)$ being equivalences $G \times G \to G \times G$. If G is grouplike, then we may form the commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{\ \Delta\ } & G \times G & \xleftarrow{\ \Delta\ } & G \\ \big\| {\scriptstyle =} & & \big\downarrow {\scriptstyle (g, gh^{-1})} & & \big\| {\scriptstyle =} \\ G & \xrightarrow[g \mapsto (g,e)]{} & G \times G & \xleftarrow[g \mapsto (g,e)]{} & G \end{array}$$

This diagram is commutative and the middle vertical map is an isomorphism (by assumption), hence it induces an equivalence on pullbacks.

Proposition IV.3 (Shearing)**.** If $G \in \mathsf{Mon(C)}^{\mathrm{gp}}$, then we get an equivalence $LG \xrightarrow{\sim} G \times \Omega_e G$.

Example IV.1. The grouplike monoids in Spc are the grouplike $\mathbb{E}_1$-spaces, which are precisely the based loop spaces. Hence, given a based space (X, x_0), there is a splitting

$$LX \xrightarrow{\sim} \Omega X \times \Omega^2 X.$$

Remark IV.1. Shearing in Proposition IV.3 is an equivalence of *pointed objects*. It is not an equivalence of monoids. (*A priori*, LX does not even admit a monoid structure. Taking $X = BG$ for a non-abelian group G is an example where LX admits an $\mathbb{E}_1$-monoid structure but does not split as one.)

Now I want to state coshearing. Shearing says that if X is a grouplike monoid, then we can "pull out X" from LX, i.e. we can write $LX = \lim(X \to X \times X \leftarrow X) = X \times \lim(* \to X \leftarrow *) = X \times \Omega X$. This argument was formal, and we want to formally dualize it to say that if X is (a cogroup?), then we can "pull out X" to get $\operatorname{colim}(X \leftarrow X \sqcup X \to X) = X \sqcup \operatorname{colim}(* \leftarrow X \to *) = X \sqcup \Sigma X$.

Let me first carefully phrase what happens for shearing. If C has finite products, then each $X \in (\mathsf{C}, \times)$ has a unique comonoid structure, namely $(X, X \to *, \Delta : X \to X \times X)$. If C has pullbacks, then we define $LX := X \times_\Delta X$. For any point $p : * \to X$, we define the constant map $c_p : X \to * \to X$ and form $\Omega(X, p) := X \times_p X$. We are wondering when we have an equivalence

$$X \times_\Delta X \overset{?}{=} X \times \left(X \times_p X\right).$$

Both sides are pullbacks (the right hand side is a pullback $X \times_{\mathrm{id}_X \times c_p} X$). So, it suffices to find an equivalence of cospans such that the following diagram commutes.

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X \times X & \xleftarrow{\Delta} & X \\ \Big| & & \Big\downarrow{\scriptstyle \cong ?} & & \Big| \\ X & \xrightarrow{\mathrm{id} \times c_p} & X \times X & \xleftarrow{\mathrm{id} \times c_p} & X \end{array}$$

By definition, maps $X \times X \to X \times X$ are precisely pairs of maps (m_1, m_2) where $m_i : X \times X \to X$. Suppose that $m = (m_1, m_2)$ makes this diagram commute. Commutativity is equivalent to two equalities:

$$\mathrm{id}_X = \pi_1(m\Delta) = m_1 \Delta \quad \text{and}$$
$$c_p = \pi_2(m\Delta) = m_2 \Delta.$$

I think we should interpret these identities as saying "m acts by $(g, h) \mapsto (g, gh^{-1})$." It is not clear to me how to precisely express this nor whether that's important. This is a sort of *shear map*; it is an equivalence if and only if X is *grouplike*. To summarize:

(1) Once C has finite products and pullbacks, we can define the relevant objects: (grouplike) monoids, LX, and $\Omega_p X$. I remark that these are defined from the unique comonoid structure on X.

(2) Given $X \in \mathsf{C}_*$, we notice that to exhibit $LX \cong X \times \Omega X$, it suffices to exhibit a map $X \times X \to X \times X$ that is also a map (equivalence) of cospans as above.

(3) We ask what this implies. We observe that commutativity of this diagram is equivalent to the existence of a "shear map" which we *morally* think of as $(g, h) \mapsto (g, gh^{-1})$. This begets a pointed comparison $LX \to X \times \Omega_p X$ that is an equivalence if and only if shearing is an isomorphism.

(4) Thus, we can say: *if X has shearing and it is an isomorphism, then we get a pointed splitting* $LX \xrightarrow{\sim} X \times \Omega_p X$. Since grouplike monoids have a shear equivalence, we have in particular a pointed splitting for $X \in \mathsf{Mon(C)}^{\mathrm{gp}}$.

Remark IV.2. In (3), we are led to ponder the existence of a shear map $X \times X \to X \times X$. (Actually, the inverse to a shear map, since it uses the inverse.) To me, this thing only comes up when X is a monoid or group—but notice we have not assumed either! I *want* to say that if $m : X \times X \to X \times X$ is an isomorphism making the diagram commute, then it encodes a grouplike monoid structure on X, but I was unable to unwind all this. If this statement is false, then maybe it is interesting to study "objects with shearing."

Think about

Now, suppose that C is an ∞-category with finite coproducts. Every object X has a unique monoid structure in $(\mathsf{C}, \sqcup)$, namely with the maps $X \sqcup X \to X$ and $\varnothing \to X$. We can dualize the definition of the free loop space here, which I will daringly call (for lack of standard terminology) the *Hochschild* of X

$$\mathrm{H}X := \mathrm{colim}(X \leftarrow X \sqcup X \to X).$$

Proposition IV.4. The Hochschild of X is the colimit of the constant S^1-shaped diagram

$$\mathrm{H}X = \mathrm{colim}\left(S^1 \to \{X\} \to \mathsf{C}\right).$$

Proof. In the ∞-category of spaces, we have $S^1 = \Sigma S^0 = * \sqcup_{* \sqcup *} *$, whence we can compute $\mathrm{colim}(S^1 \mapsto X) = X \sqcup_{X \sqcup X} X = \mathrm{H}X$. $\square$

Remark IV.3. A somewhat standard and classical notation is to write $S^1 \otimes X$ for the constant S^1-shaped colimit on X. (This works more generally for any simplicial set K.) Somehow this has become unfashionable?

But now we run into a subtlety. Shearing was about "pulling out X" to obtain

$$LX = \lim(X \to X \times X \leftarrow X) = X \times \lim(* \to X \leftarrow *) = X \times \Omega X.$$

Then "coshearing" should establish something like $\mathrm{H}X = X \sqcup \mathrm{colim}(\varnothing \leftarrow X \to \varnothing)$. This gives pause: in many examples (e.g. $\mathsf{C} = \mathsf{Set}$ or Spc) the initial object $\varnothing$ is *strict* (no maps to it exist), in which case there is no hope of making sense of the righthand thing. But all is not lost—interesting and important stuff happens in categories where this is *not* an issue! One example is the motivating example for this post, the ∞-category $\mathsf{CMon}(\mathsf{Sp}) = \mathsf{Mon}_{\mathbb{E}_\infty}(\mathsf{Sp})$ of $\mathbb{E}_\infty$-ring spectra. Actually, something rather high-powered and fundamental occurs here: commutative ring spectra form a 0-*semiadditive* category. For one reason or another, I think these are good categories within which to think about Hochschildren $\mathrm{H}X$.

Figure out what conditions on X grant us a splitting $\mathrm{H}X = X \oplus \mathrm{colim}(0 \leftarrow X \to 0)$. Want these conditions to hold in the case that $X = \mathbb{S}^{BA}$.

We wrap up by applying this theory to study the topological Hochschild homology of $\mathbb{E}_\infty$-ring spectra. This is an important step in §3 of [Bur+23]. For this, we fix a prime p and work p-typically and p-completely. We will be studying the *spherical cochains functor*

$$\mathbb{S}^{(-)} : \mathsf{Spc} \to \mathsf{CAlg}(\mathsf{Sp}_p)^{\mathrm{op}} \quad \text{given by} \quad X \mapsto \mathrm{Map}(\Sigma^\infty_+ X, \mathbb{S}_p^\wedge).$$

We will also make use of the *spherical Witt vectors* construction and a recognition lemma.

Proposition IV.5. There is an adjunction

$$\mathrm{Perf}_{\mathbb{F}_p} \underset{\pi_0^\flat}{\overset{\mathbb{W}}{\rightleftarrows}} \mathsf{CAlg}(\mathsf{Sp}_p)$$

between perfect $\mathbb{F}_p$-algebras and p-complete commutative algebras. This adjunction satisfies the following.

(1) The right adjoint $\pi_0^\flat(R)$ is computed as the inverse limit along the Frobenius on $\pi_0(R)/p$.

(2) The spherical Witt vector construction $\mathbb{W}(-)$ is fully faithful. Its image is spanned by those connective R such that $R \otimes \mathbb{F}_p$ is discrete and perfect, in which case $R \cong \mathbb{W}(R \otimes \mathbb{F}_p)$.

We will use Witt vectors (and their recognition lemma) to model spherical cochains in a concrete, manipulable manner. Namely, we will model spherical cochains by "actual cochains." For this, we consider a discrete, finite rank, projective $\mathbb{Z}_p$-module A. The module structure encodes a p-adic topology, thus we may consider its continuous $\mathbb{F}_p$-cochains $C^0(A) := \mathrm{Map}(A, \mathbb{F}_p)$. We also consider $A^\delta := A$ with the discrete topology and observe that $C^0(A^\delta)$ is the set of all functions $A \to \mathbb{F}_p$.

Lemma IV.1. The counit fashions an identification of commutative algebras $\mathbb{W}C^0(A^\delta) \xrightarrow{\sim} \mathbb{S}^{\Omega BA}$.

Proof. Note that ΩBA is just A regarded as a discrete space, whence $\mathbb{S}^{\Omega BA} = \prod_A \mathbb{S}_p^\wedge$. Now we check

$$\mathbb{F}_p \otimes \mathbb{S}^{\Omega BA} = \prod_A (\mathbb{F}_p \otimes \mathbb{S}_p^\wedge) = \prod_A \mathbb{F}_p.$$

Here we used that $\mathbb{F}_p \otimes -$ commutes with products that are bounded below. Since $\prod \mathbb{F}_p$ is discrete and perfect, the recognition lemma yields an identification $\mathbb{W}(\prod_A \mathbb{F}_p) \xrightarrow{\sim} \mathbb{S}^{\Omega BA}$ via the counit. This is the desired equivalence since by definition we have $\prod_A \mathbb{F}_p = C^0(A^\delta)$. □

Lemma IV.2 (Lemma 3.4 in [Bur+23])**.** In fact, applying the counit to the canonical assembly map yields an identification of commutative algebras

How do we actually get this map? What a pain

$$\begin{array}{ccc} \mathbb{W}C^0(A) & \xrightarrow{i} & \mathbb{W}C^0(A^\delta) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S} & \longrightarrow & \mathbb{S}^{\Omega BA} \end{array}$$

Proof. The argument is identical to that used to prove the previous lemma, but now the computation is a bit harder, and one must verify commutativity. The cited lemma comes with a proof. □

Now we can do something fun. For a finite, projective $\mathbb{Z}_p$-module A, the classifying space $BA \in \mathsf{Spc}$ is a grouplike monoid, thus shearing begets a pointed splitting $LBA \xrightarrow{\sim} BA \times \Omega BA$. We can "push shearing through $\mathbb{S}^{(-)}$" and get

Is this true, or are we thinking in the category $\mathsf{Spc}_{/BA}$ of spaces over BA?

$$\mathbb{S}^{BA \times \Omega BA} \xrightarrow{\sim} \mathbb{S}^{LBA}.$$

Alternatively, we can coshear grouplike comonoids in $\mathsf{CAlg}(\mathsf{Sp}_p)$. Being a category of commutative monoid objects, this category has tensor products given by coproducts. Since $\mathbb{S}^{(-)}$ preserves products, it preserves grouplike monoids, so in particular we can coshear $\mathbb{S}^{BA}$ and get

$$\mathbb{S}^{BA} \otimes (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) \xrightarrow{\sim} \mathrm{THH}(\mathbb{S}^{BA}).$$

These splittings are related by *assembly maps*. These are formal; given composable functors F and G, there is a canonical map $\lim(GF) \to F \lim(G)$ assuming the appropriate limits exist. In our case, we recall that in CAlg, the tensor product is a coproduct and that THH is computed as a constant S^1-shaped colimit (i.e., $HR = \mathrm{THH}(R)$). Then our assembly maps give us the following commutative diagram.

$$\begin{array}{ccc} \mathbb{S}^{BA} \otimes (\mathbb{S} \otimes_{\mathbb{S}^{BA}} \mathbb{S}) & \longrightarrow & \mathbb{S}^{BA \times \Omega BA} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{THH}(\mathbb{S}^{BA}) & \longrightarrow & \mathbb{S}^{LBA} \end{array} \qquad (1)$$

This is somehow a canonical comparison between $\mathbb{S}^{BA}$-coshearing (the left-hand side) and $\mathbb{S}^{(-)}$ applied to BA-shearing (the right-hand side). Combined with the previous lemma, we get the following.

Lemma IV.3 (Lemma 3.6 in [Bur+23])**.** We have a natural identification of commutative algebras

$$\begin{array}{ccc} \mathbb{S}^{BA} \otimes \mathbb{W}C^0(A) & \longrightarrow & \mathbb{S}^{BA} \otimes \mathbb{W}C^0(A^\delta) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{THH}(\mathbb{S}^{BA}) & \longrightarrow & \mathbb{S}^{LBA} \end{array}$$

Remark IV.4. In place of $\mathbb{S}^{(-)}$, we may consider an arbitrary product-preserving presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and study (co?)assembly as a measure of how F fails to commute with shearing, as in Eq. (1). Is this an interesting thing to do? What are examples of F for which the S^1-(co)assembly is always an equivalence? This fails for $F = \mathbb{S}^{(-)}$, in fact it fails at the object $B\mathbb{Z}$, which is the main event of [Bur+23, §3]. Does the full subcategory spanned by these presheaves have a characterization?

Wrap up by talking about how this is helpful: decompose the fiber of THH-coassembly.

IV.2 (4/9) Semiadditivity I

I thought and wrote about semiadditivity quite a bit in my previous notebook. I will try to proactively compartmentalize my thoughts by declaring this to be the first in a series of posts (in this notebook) that will include all things semiadditive.

Here's the elevator pitch.

On floor -1, we consider the property of *pointedness*. This refers to categories with a zero object, i.e. an object 0 that is both initial and terminal. Pointedness amounts to (1) the *existence* of initial and terminal objects, and (2) the *existence* of an isomorphism between them. Suppose that C has initial/final objects; there already exists a unique map

$$\varnothing_{\mathsf{C}} \to *_{\mathsf{C}} .$$

This map exists and is unique because $\varnothing_{\mathsf{C}} = \operatorname{colim} \varnothing$ and $*_{\mathsf{C}} = \lim \varnothing$. Furthermore, it is an isomorphism once any map $*_{\mathsf{C}} \to \varnothing_{\mathsf{C}}$ exists (which becomes its inverse) since $\varnothing_{\mathsf{C}}$ and $*_{\mathsf{C}}$ are without nontrivial endomorphisms. Pointedness has pleasant consequences. Notably, we have a distinguished *zero map* $A \to 0 \to B$ between any two objects. This enriches C in Set_*.

On floor 0, we suppose C is pointed and consider the property of *semiadditivity* (aka *having direct sums* or *having biproducts*). Morally, direct sums should be a coherent way to "add" objects. Formally, given $A, B \in \mathsf{C}$ this amounts to (1) the *existence* of their product and coproduct, and (2) the *existence* of an inverse to the unique *identity matrix* map $A \sqcup B \to A \times B$. What is this map? That would be another consequence of pointedness—it is the map

$$\begin{pmatrix} \mathrm{id}_A & 0_{A,B} \\ 0_{B,A} & \mathrm{id}_B \end{pmatrix} : A \sqcup B \to A \times B.$$

See that "having direct sums" is a *property* because no choices are involved in constructing this map, rather it exists by virtue of pointedness (which gives us zero maps). Having direct sums has pleasant consequences. Notably, given two maps $f, g : A \to B$, we have a distinguished map

$$f + g := \left(X \xrightarrow{f \times g} Y \times Y \xrightarrow{\sim} Y \sqcup Y \to Y \right).$$

This enriches C in commutative monoids.

Let's meditate out loud and try to extract an inductive definition:

- The objects $\varnothing_{\mathsf{C}}$ and $*_{\mathsf{C}}$ are a limit and colimit over the *empty functor*, respectively. The category C is *pointed* as soon as there exists any map $*_{\mathsf{C}} \to \varnothing_{\mathsf{C}}$ which automatically becomes the unique isomorphism of the zero object. We suggestively write it $\mathrm{Nm}_{(-1)}$. Given $A, B \in \mathsf{C}$ this allows us to define a canonical *zero map*

$$A \to *_{\mathsf{C}} \xrightarrow{\mathrm{Nm}_{(-1)}^{-1}} \varnothing_{\mathsf{C}} \to B.$$

If C is pointed, then we can use the zero maps to construct an "identity matrix" map between finite (co)products (when the relevant (co)products exist), which we suggestively denote

$$\mathrm{Nm}_{(0)} : \coprod_{i=1}^{N<\infty} A_i \to \prod_{i=1}^{N<\infty} A_i.$$

- A finite (co)product is a (co)limit over a *diagram* $F : X \to \mathsf{C}$ *such that* X *is a finite, discrete set*. Such a set is equivalently a *space with finite* π_0 *and no higher homotopy groups.* If C is pointed, then there is a canonical map $\mathrm{Nm}_{(0)} : \mathrm{colim} F \to \lim F$ by the above, but it is not necessarily an isomorphism. Just as pointedness describes when $\mathrm{Nm}_{(-1)}$ is an equivalence (which degenerates to its existence), we will say C is *semiadditive* if $\mathrm{Nm}_{(0)}$ is an isomorphism for every finite set X. In this case, just as we use $\mathrm{Nm}_{(-1)}^{-1}$ to get zero maps (i.e., a Set_*-enrichment), we can use $\mathrm{Nm}_{(0)}^{-1}$ to get more structure (namely, a CMon-enrichment). Got lazy... finish this?

This is what I would call the "norms and integration" approach. The enrichment granted by m-semiadditivity is treated as an important *consequence*. But in fact, this enrichment structure is somehow unique, i.e. it is a *property*, and it can be used to characterize m-semiadditivity. This is what I would call the "higher monoids" approach. That m-semiadditivity is both a *structure* and a *property* is abstracted by the theory of *modes*.

IV.3 (4/12) Semiadditivity II — Basic language and calculus of local systems

I want to explain how semiadditivity is related to power operations. Today I will fresh up on local systems and the basic "calculus" powered by semiadditivity.

Recall the examples motivating the semiadditive formalism: certain functors $X : A \to \mathsf{Sp}$ come with canonical *norm maps* $\mathrm{Nm}_X : \mathrm{colim} X \to \lim X$ which behave nicely and suggest some rich, large-scale architecture upon Sp. The most basic case is when $A = \varnothing$, in which case the norm is an isomorphism of initial and terminal objects, which amounts to pointedness (which also follows from stability). The second most basic case is when X is a finite set, in which case Nm_X is the "identity matrix" map $\coprod_A X_a \to \prod_A X_a$ built from identity and zero maps. Here, the norm is again an isomorphism, and this amounts to Sp admitting *finite direct sums* (which also follows from stability). The next most basic case is a remarkable theorem: if X is 1-finite, then we have an associated norm $\mathrm{Nm}_X : \mathrm{colim} X \to \lim X$ and it is an isomorphism. (The theorem I refer to is Tate vanishing for finite group actions, i.e. for functors $X : BG \to \mathsf{Sp}$, which essentially handles the statement here.)

There are two problems here: constructing Nm_X and figuring out when it is an isomorphism. See that $X \mapsto \mathrm{colim} X$ and $X \mapsto \lim X$ extend to an adjoint triple

$$\mathsf{Sp} = \mathrm{Fun}(*, \mathsf{Sp}) \underset{\underset{p_* = \lim(-)}{\longleftarrow}}{\overset{\overset{p_! = \mathrm{colim}(-)}{\longleftarrow}}{\xrightarrow{\;\perp\; p^* \;\perp\;}}} \mathrm{Fun}(X, \mathsf{Sp})$$

Here, $p : X \to *$ denotes the projection and $p^* : \mathsf{Sp}^X \to \mathsf{Sp}$ is the *pullback* morphism defined by precomposition. More generally, we can "relativize" the situation and consider an arbitrary map of spaces $q : A \to B$ and ask about an adjoint system

$$q_! \dashv q^* \dashv q_*.$$

In particular, we ask: when do these adjoints exist? How can we construct Nm_q? And when is Nm_q an equivalence? Here is the basic investigation out of which ambidexterity and semiadditivity arise.

Given $q : A \to B$ and C a category, the problem of exhibiting adjoints $q_! \dashv q^* \dashv q_*$ to the pullback $q^* : B^{\mathsf{C}} \to A^{\mathsf{C}}$ is purely formal. This is because left/right adjoints are necessarily computed by "fiberwise" colimit/limits. Hence, if C admits all (co)limits indexed by (co)fibers of q, then $q_!$ and q_* exist. In the business, we say C *has q-limits and q-colimits* when this is the case.

Constructing Nm_q is trickier; I will breeze over this point, c.f. [CSY18, §3.1] for details. First, one considers the diagonal pullback square for q and observes that the induced square of C-valued local systems satisfies the *$BC_!$ and BC_* conditions.* (This is true for all pullback squares in Spc.) Whatever this means, it gives us a map $\beta_!^{-1} : q^* q_! \to (\pi_2)_! \pi_1^*$. Second, we suppose as given a natural isomorphism $\mathrm{Nm}_\delta : \delta_! \xrightarrow{\sim} \delta_*$, which corresponds (c.f. [CSY18, §2.1]) to a *wrong-way unit* $\mu_\delta : \mathrm{id} \to \delta_! \delta^*$. Now we may form

$$\nu_q : q^* q_! \xrightarrow{\beta_!^{-1}} (\pi_2)_! \pi_1^* \xrightarrow{\mu_\delta} (\pi_2)_! \delta_! \delta^* \pi_1^* \cong \mathrm{id}.$$

We define $\mathrm{Nm}_q : q_! \to q_*$ as the mate of ν_q under $q^* \dashv q_*$.

The essential observation now is that if q is m-truncated, then δ_q is $(m-1)$-truncated. With that, by the above process, if we can supply norm *isomorphisms* for all $(m-1)$-truncated maps, then we get a system of norms for all m-truncated maps (which are not necessarily isomorphisms).

Definition IV.2 ([CSY18] Def 3.1.5)**.** Given C and a map $q : A \to B$ of spaces, define (weak) ambidexterity and canonical norms.

Definition IV.3 ([CSY18] Def 3.1.10)**.** Define m-semiadditivity.

That establishes the basic language—norms, ambidexterity, and semiadditivity. Next, I would like to explore the *integration* procedure made possible via norms. This integration is a *higher addition* of sorts, and as such it is intimately related to semiadditivity. Much of this calculus is actually possible given any abstract *normed functor* [CSY18, §2]. They refer to calculus in this generalized situation as *axiomatic integration*. The motto is, "iso-norms permit integration."

Let us first work in the case of local systems (i.e., the non-axiomatic case). This means that we fix a map of spaces $q : A \to B$ and a category C. Suppose that q is m-finite, that C is $(m-1)$-semiadditive, and that C admits q-(co)limits. These assumptions guarantee that $\mathrm{Nm}_q : q_! \to q_*$ exist.

Definition IV.4. In the above situation, define the *wrong-way counit* as the mate $\nu : q^* q_! \to \mathrm{id}$ of Nm_q.

Recall that *a priori* our adjunctions are oriented $q_! \dashv q^* \dashv q_*$. *We do not necessarily have $q^* \dashv q_!$.* In particular, the mate ν is not necessarily a counit witnessing this adjunction, hence the "wrong-way" terminology. On account of ν "mixing up" $q_!$ and q_*, if ν *does* realize $q^* \dashv q_!$, we should like to regard this as equivalent to Nm_q identifying $q_!$ and q_*. That was a long-winded way to say the following.

Proposition IV.6 ([CSY18] Lemma 2.1.6)**.** Given $Y \in \mathsf{C}^A$, the norm $\mathrm{Nm}_q : q_! \to q_*$ is an isomorphism at Y if and only if the mate $\nu : q^* q_! \to \mathrm{id}$ is a counit map at Y.

We therefore get two equivalent perspectives, that of the norm $\mathrm{Nm}_q : q_! \to q_*$ and its wrong-way counit $\nu_q : q^* q_! \to \mathrm{id}$, and Nm_q is an isomorphism if and only if ν_q is an actual counit. Conceptually, this (to me) affirms the obtuse construction of Nm_q by characterizing it in familiar terms. It also lets you make alternate definitions or constructions typically involving the norm. For instance, we can now define *integration* in two ways.

Definition IV.5 ([CSY18] Def 2.1.11)**.** Let $q : A \to B$ be an m-finite map and suppose that C is $(m-1)$-semiadditive and that C admits q-(co)limits. Then for every pair $X, Y \in \mathrm{Fun}(B, \mathsf{C})$, we define an *integration map*

$$\int_q : \mathrm{Map}_{\mathrm{Fun}(A,\mathsf{C})}(q^* X, q^* Y) \to \mathrm{Map}_{\mathrm{Fun}(B,\mathsf{C})}(X, Y)$$

as the composition

$$\mathrm{Map}_{\mathrm{Fun}(A,\mathsf{C})}(q^*X, q^*Y) \xrightarrow{q_*} \mathrm{Map}_{\mathrm{Fun}(B,\mathsf{C})}(q_*q^*X, q_*q^*Y) \xrightarrow{\mathrm{Nm}_q^{-1}} \mathrm{Map}_{\mathrm{Fun}(B,\mathsf{C})}(q_*q^*X, q_!q^*Y) \xrightarrow{c_! \circ - \circ u_*} \mathrm{Map}_{\mathrm{Fun}(B,\mathsf{C})}(X, Y).$$

See that this is equivalent to the composition (using the *wrong-way unit* $\mu_q : \mathrm{id} \to q_!q^*$)

$$\mathrm{Map}_{\mathrm{Fun}(A,\mathsf{C})}(q^*X, q^*Y) \xrightarrow{q_!} \mathrm{Map}_{\mathrm{Fun}(B,\mathsf{C})}(q_!q^*X, q_!q^*Y) \xrightarrow{c_! \circ - \circ \mu} \mathrm{Map}_{\mathrm{Fun}(A,\mathsf{C})}(X, Y).$$

Definition IV.6. Given $X : B \to \mathsf{C}$ in the situation above, we define $|q|_X : X \to X$ as the integral

$$|q|_X := \int_q q^*\mathrm{id}_X = \int_q \mathrm{id}_{q^*X}.$$

Remark IV.5. The composite $c_!^q \circ \mu_q$ is an endomorphism $\mathrm{id}_{\mathrm{Fun}(B,\mathsf{C})} \to \mathrm{id}_{\mathrm{Fun}(B,\mathsf{C})}$ with components $|q|_X$. We may therefore write $|q| = c_!^q \circ \mu_q$. If $B = *$ and so $q : A \to *$ is the projection, we regard $|q|$ as *multiplication by the cardinality of A* and write $|A| := |q|$.

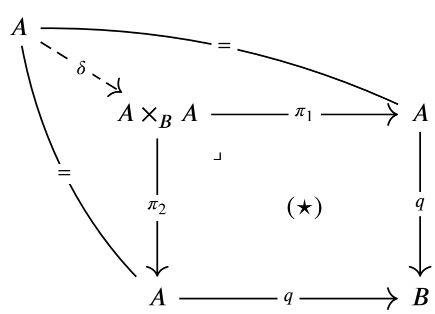
Proposition IV.7 (§2.1 [CSY18])**.** Integration satisfies a version of homogeneity and Fubini's theorem.

(Talk about additional properties of integation of local systems. In particular, its functoriality over base-change squares induced by pullbacks of spaces, c.f. [CSY18, §3.1]. Note the need for ambidextrous squares, Beck-Chevalley conditions. Derive distributivity and additivity.)

IV.4 (4/16) Semiadditivity III, Squares

The definition of norms for local systems is not very hard, but I do not think it is intuitive. I am pleased to have learned the "wrong-way (co)unit" perspective, which is not deep, because I think it gives an interpretation of Nm_q that is somewhat intuitive. Today I want to review how [CSY18] distills norms for local systems into a general categorical procedure.

Let me first highlight some ingredients for constructing norms for local systems. We suppose as given an m-truncated map $q : A \to B$ and C admitting q-(co)limits, so that $q_!$ and q_* exist. Our approach to constructing Nm_q takes advantage of basic special properties of δ_q. Namely, it is $(m-1)$-truncated, and it arises as the induced map in the following diagram.



What's special is that the square $(\star)$ induces a *Beck-Chevalley square* [CSY18, §2.2] after taking $\mathsf{C}^{(-)}$. Generally, squares of functors with adjoints (more precisely, maps of normed functors) come with *left and right Beck-Chevalley transformations* $\beta_!, \beta_*$ measuring commutativity of taking adjoints; by a *Beck-Chevalley square*, I mean one for which these $\beta_!, \beta_*$ are equivalences. In our case, this yields a canonical map $\beta_!^{-1} : q^*q_! \to \pi_{2!}\pi_1^*$. We want a "wrong-way counit" $q^*q_! \to \mathrm{id}$, why not just define it as this "anti-equivalence" $\beta_!^{-1}$ composed with a "wrong-way unit" $\mathrm{id} \to \delta_!\delta^*$? That is precisely what we do: suppose we have a unit map $\mu_\delta : \mathrm{id} \to \delta_!\delta^*$ (equivalently, that we have an equivalence Nm_δ) and compose

$$q^*q_! \to \pi_{2!}\pi_1^* \to \pi_{2!}\delta_!\delta^*\pi_1^* \xrightarrow{\sim} \mathrm{id}.$$

Note that the last equivalence *is a direct consequence of the definition of* δ (using the commutative diagram). This is how we defined the norm and hence integration. We can work more generally with *normed functors and maps between them*, think about *Beck-Chevalley properties* for the squares arising from such maps, and study norms and integration in this situation (ultimately finding consequences for the local systems case).

do that

IV.5 (4/20) The equivariant Θ^p power operation

If C is a symmetric monoidal ∞-category, then the p-th power map $\mathsf{C} \to \mathsf{C}$ naturally lifts to a functor $\Theta^p\colon \mathsf{C} \to \mathrm{Fun}(BC_p, \mathsf{C})$. This functor takes p-th powers with their cyclic action. More generally, for a space A we may construct a functor $\Theta^p_A : \mathrm{Fun}(A, \mathsf{C}) \to \mathrm{Fun}(A \wr C_p, \mathsf{C})$ as the composite

$$\mathrm{Fun}(A,\mathsf{C}) \longrightarrow \mathrm{Fun}(A \wr C_p, \mathsf{C} \wr C_p) \xrightarrow{C_p \hookrightarrow \Sigma_p} \mathrm{Fun}(A \wr C_p, \mathsf{C}^p_{h\Sigma_p}) \xrightarrow{\otimes} \mathrm{Fun}(A \wr C_p, \mathsf{C}).$$

Taking this relative perspective, we can get some handle on Θ^p, namely we can prove functoriality [CSY18, Lemma 3.4.3] and concretely express Θ^p via integration (when integration is possible).

Let's just assume functoriality of Θ^p with respect to maps of spaces. Then for a map $q : A \to B$ of spaces, we get a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(B,\mathsf{C}) & \xrightarrow{\Theta^p_B} & \mathrm{Fun}(B \wr C_p, \mathsf{C}) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (q \wr C_p)^*} \\ \mathrm{Fun}(A,\mathsf{C}) & \xrightarrow{\Theta^p_A} & \mathrm{Fun}(A \wr C_p, \mathsf{C}) \end{array}$$

Now, we can make formal observations: (I) if q is m-finite, then $q \wr C_p$ is m-finite; (II) if C is $(m-1)$-semiadditive and admits q-(co)limits, then q and $q \wr C_p$ are *normed* and the above square is a *map of normed functors* aka *normed square*; and (III) if in addition $\otimes_{\mathsf{C}}$ distributes over q-(co)limits, then this square is *Beck-Chevalley* [CSY18, Lemma 3.4.3]. All this lets you prove that when these normed functors possess integration (i.e. are iso-normed), this equivariant power operation $\Theta^p_{(-)}$ commutes with integration. Precisely:

Proposition IV.8 (Prop 3.4.6 and Thm 3.4.8 [CSY18])**.** If C is m-semiadditive and $\otimes_{\mathsf{C}}$ commutes with m-finite colimits, then for any m-finite map q, the resulting Θ^p square is ambidextrous. In particular,

$$\Theta^p_B\Big(\int_q f\Big) = \int_{q \wr C_p} \Theta^p_A(f).$$

V September 2024

V.1 (9/15) Categorical Descent I (redux) — rambling, bundles as an example of a stack

> [Sheaf theory] is an octopus spreading itself throughout everyone else's history.
>
> *John Gray in "Fragments of the History of Sheaf Theory"*

I took a sabbatical to move to Boston, study for my quals, etc. Now, our year has started and Tommy is teaching a course on motivic homotopy theory. To start, we are analyzing a baby case of descent and using this to motivate the introduction of simplicial methods to algebraic geometry (toward the category of motivic spaces). This is quite nice and cool. Furthermore, it brings me back to a circle of ideas I tried to discern previously, namely descent in homotopy/category theory. I did not make progress toward understanding any of this last time, but now that I feel less pressure (to graduate, apply to grad school, move, ...) I want to try again. As a non-algebraic geometer, this is also my way of learning some basic algebaic geometry, what descent even is, and a lot of other algebra I somehow missed as an undergraduate.

Some references (many old, some new):

1. HTT, in particular §6.1.
2. Tommy's motivic course;
3. This n-Café post and the links therein;
4. Milne's course notes on descent;
5. Joshua Ruiter's notes on descent;
6. The series of papers "Facets of descent" by Janelidze and Tholen;
7. Mesablishvili's paper "Comonadicity of the extension-of-scalars functor" here;
8. Marc Paul Noordman's bachelor thesis, an easily accessed reference for principal G-bundles *over spaces* and their relationship to Čech cocycles;
9. The references from the last time I tried to learn descent.

What's descent? We are often interested in an object X and (often geometric) things $G(X)$ living in/over/around X. We may try taking a simpler object Y and look at the same type of things $G(Y)$ living over Y. In the situation that we can descend/restrict our structures $G(X) \to G(Y)$, *descent* asks what information is lost. Then we seek to express

$$G(X) = G(Y) + \text{ descent data.}$$

The challenge is to coherently express the descent data, as well as its pairing with $G(Y)$. The goal is to develop strong enough language to say that descent is a *property* of a presheaf to fulfill with respect to a cover. In my next few posts, I want to tour examples of descent in as (self-)instructive a manner as possible. Today I want to explain basic topological examples and find suggestions toward greater theory: cohomology, stacks, homotopy theory, ...

Example V.1. We consider the simplest example of a sheaf of sets F on a space X. Given two sections s_U, s_V defined on an open cover $U \cup V = X$, if s_U, s_V agree on $U \cap V$, then their restrictions are equal in $F(U \cap V)$. Whether or not their restrictions are equal in the set $F(U \cap V)$ precisely detects whether s_U, s_V arise by restricting a global section $s \in F(U)$, which we ask to be unique; this is the sheaf condition for the cover $U \cup V = X$. Another way to phrase this (for a general open cover) is that the diagram

$$F(X) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_X U_j)$$

is an equalizer, i.e. realizes $F(X)$ as the pullback of the indicated cospan. Notice that the sheaf condition amounts to asking F to *satisfy descent* in the sense that, for any open cover $(U_i \to X)$, we have

$$F(X) = \prod F(U_i) + \underbrace{\text{descent data}}_{\text{knowing if } s_i = s_j \text{ on } U_{ij}} .$$

I highlighted the fact that $F(U \cap V)$ is a *set*. This made things easy: we measured the compatibility of s_U and s_V by restricting them to $F(U \cap V)$, and there we need only ask whether $s_U|_V$ and $s_V|_U$ are equal. This has the obfuscating consequence of turning our descent data into a *property*. If F is valued in categories, the situation is not so straightforward. In this case, our sections s_U, s_V may become *isomorphic* objects in $F(U \cap V)$, but there may be many different isomorphisms. Is the choice of isomorphism important? Consider the example.

Example V.2. Consider the sheaf mapping U to the category of (vector or principal) bundles over U. Then any global section restricts along a trivializing open cover to a family of trivial bundles. In particular, *any* two bundles can be decomposed over an open cover so that their local sections are isomorphic.

By this example, we understand that the choice of local isomorphism is important—if it were not, then all bundles over a space would be isomorphic! We have identified the problem: if F is a presheaf of categories, then the functor

$$F(X) \to \prod F(U_i)$$

need not reflect isomorphisms. This raises the question: is tracking the isomorphisms enough? What extra *descent data* is necessary to recover $F(X)$ from the $F(U_i)$? How can we functorially express this?

Example V.3 (Bundles are a 2-sheaf, rewrite)**.** I'm going to be annoying and consider the most general relevant example I can think of. Consider the weakest definition of *bundle*, that being nothing more than a continuous map $E \to X$. This is limp, but will give us our most unobstructed view of nontrivial descent. In consideration is the presheaf

$$bun : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cat}$$

acting by $X \mapsto \mathsf{Top}_{/X}$ and $f \mapsto f^*$. We can ask: *is bun a sheaf?* This should mean that for any open cover $(U_i \to X)$, the global $bun(X)$ is somehow coherently recovered from the local $bun(U_i)$ and their gluing data. The "somehow" here is nebulous. Let's be precise and ask: for an open cover $\{u_i : U_i \hookrightarrow X\}_i$, is the diagram

$$bun(X) \to \prod bun(U_i) \rightrightarrows \prod bun(U_{ij})$$

an equalizer? That is, does the first functor witness $bun(X)$ as a pullback? Here, something new happens—we are forced to acknowledge that Cat is a 2-category and that we are handling a 2-pullback. These possess more data than ordinary pullbacks: e.g. the objects of our 2-pullback consist of triples $(E_i \to U_i, E_j \to U_j, \phi_{ij} : E_{ij} \xrightarrow{\sim} E_{ji})$. The most essential data contained are the homeomorphisms (ϕ_{ij}); if we denote for every i, j, k the isomorphisms $\phi_{ijk} : E_{ijk} \xrightarrow{\sim} E_{kji}$, $\phi_{kij} : E_{kij} \xrightarrow{\sim} E_{jki}$, and $\phi_{jki} : E_{jki} \to E_{ijk}$, the 2-pullback structure turns out to force a *cocycle condition*

$$\phi_{ijk} = \phi_{kij} \circ \phi_{jki}.$$

It is clear that given $E_i \to U_i$ with identifications (ϕ_{ij}) satisfying the above conditions, we can re/construct $E \to X$ which restricts back to the $E_i \to U_i$. Fact: the 2-pullback is totally recovered by this data. Thus, our *bun* satisfies descent for open covers, in particular

$$bun(X) = \prod bun(U_i) + \underbrace{\text{descent data}}_{\text{Čech 1-cocycle of identifications on overlaps}} .$$

Example V.4. A convenient reference for 2-pullbacks is Section 003O on the Stacks Project.

Example V.5 (Fiber bundles with structure group, principal bundles)**.** One can instead take *bundles* over X to mean fiber bundles (with structure group) or principal G-bundles. By a similar line of reasoning, given an open cover $\mathcal{U}$, you can extract from each bundle $E \to X$ a 1-cocycle $\sigma_{E,\mathcal{U}}$ that encodes how the restrictions of E glue together into E. (I think we want $\mathcal{U}$ to trivialize $E \to X$.) And just as above, it turns out the converse is true: given the 1-cocycle of gluing data over U, we can construct $E \to X$ so that it is trivialized by $\mathcal{U}$ and has that cocycle as its transition data. If we're talking about fiber bundles: if the structure group acts faithfully, then "cohomologous" cocycles yield isomorphic bundles; this is the fiber bundle (re)construction theorem. If we're talking about principal G-bundles: no additional hypothesis is needed so that "cohomologous" cocycles yield the same G-bundle. Altogether, we have (vaguely, without rigorous definitions) related

$$\text{bundles} \leftrightarrow \text{1-cocycles / relation.}$$

We can make this into a precise comparison between principal G-bundles and the first Čech cohomology $H^1(\mathcal{U}; G)$. I will not go into that now. This is the origin for the cocycle language.

V.2 (9/17) Descent II — The simplicial language

Let me put yesterday into more deliberate, concise language. We consider objects X, Y where Y is somehow simpler than X, in particular with respect to some sort of geometric structures $G(X)$ and $G(Y)$ related to X and Y. We are interested in systematically expressing

$$G(X) = G(Y) + \text{ descent data.}$$

We have not yet made this formal; but when we do, we shall say something like "G satisfies descent along Y" if such an expression is possible.

Now let's make this formal. We saw that for a sheaf of sets $F : \mathsf{Op}(X)^{\mathrm{op}} \to \mathsf{Set}$, a global section s is recovered as compatible local sections s_i together with the "data" that for any two s_i, s_j we have $s_{ij} = s_{ji}$, which is really the *property* that s_i, s_j agree on overlaps. That is to say, the following map is an equivalence.

$$F(X) \to \lim\left(\prod F(U_i) \rightrightarrows \prod F(U_i \times_X U_j)\right).$$

We next considered "bundles on a space." I stated things generally, considering a functor $\mathit{bun} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cat}$ that sends X to its slice $\mathsf{Top}_{/X}$. But you can fix a space X and think of this as a presheaf of categories $F : \mathsf{Op}(X)^{\mathrm{op}} \to \mathsf{Cat}$. The point was that given an open cover $\{U_i \hookrightarrow X\}$, we needed the following data to uniquely determine a global section $E \to X$:

(i) Local sections $E_i \to U_i$, and

(ii) Homeomorphisms $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ satisying the *cocycle condition* that given three U_i, U_j, U_k,

$$\phi_{ijk} = \phi_{kij}\phi_{jki}.$$

In contrast to the sheaf of sets F, the identification on overlaps is *data*—we must specify it. Bundles in fact provide examples where the global section *determines and is determined by* the choice of identifications, the local sections are not even relevant (they're all isomorphic). This relationship between $\mathit{bun}(X)$ and gluing data is equivalent to the canonical map below being an equivalence.

$$F(X) \to \lim\left(\prod F(U_i) \rightrightarrows \prod F(U_{ij})\right).$$

A key point is that the limit formulation still works because the limit is formed in Cat, thus it carries higher (= 2-categorical) data, which turns out to be precisely the gluing data.

Remark V.1. In asserting that $F(X) \to \lim(\dots)$ is an equivalence, we make a notable claim: it suffices to specify data on U_i (sections) and data on U_{ij} (identifications) that are compatible on triple overlaps, and no further data/properties must be checked (all is well on $(n > 3)$-overlaps). Geometrically, this is believable, but we point it out since we will tease it into a more general phenomenon.

To generalize, we will replace our space (X, τ) with a *site* (C, τ).

V.3 (9/20) Colimits are computed pointwise

VI October 2024

VI.1 (10/5) Duality and the fertile crescent

I think it's right to say that the fertile crescent of algebraic topology was witnessed during the 1950's and 60's in the coalescence of "classical" algebraic topology, manifold theory, and stable homotopy theory. A proper account is not my job, but broadly speaking, this period somehow witnessed an explosion of serious, formal considerations of: spectra and the stable homotopy category, bundle theory and characteristic classes, K-theory and periodicity, cobordism, operations and the cohomological perspective, and (esecially) the interactions of all the above.

Duality is one main character here. Poincaré observed and used his eponymous duality from the outset of his work (hence, the start of algebraic topology), and duality phenomena were further studied and sophisticated in the 50's and 60's. I am actually not too familiar with the details of this story, but lately I am needing the theory more, so learning this story seems like a good and fun idea. I want to write just a bit down for memory's sake.

Poincaré first suggested his eponymous duality theorem in 1893, to him a duality in Betti numbers b_i. This was in an effort to prove that a closed, oriented, odd-dimensional manifold has zero Euler characteristic. My impression (having not read the original manuscript) is that Poincaré merely stated duality as if it were obvious. He would give two proofs a couple years later in *Analysis situs*, and I think a "more correct" proof in 1900.

It is worth remarking on the mathematical environment Poincaré existed in, which would be quite alien to us. Firstly, Betti numbers were an invariant attached to a finite triangulation or cell complex,[14] whence much ancestral work in this area concerned analysis and manipulations of these structures.[15] For example, one of Poincaré's original (correct?) proofs of duality argued by a *dualizing triangulation* construction, showing that it had complementary Betti numbers, yet shared a barycentric subdivision with the original triangulation. This had a certain "combinatorial" flavor, in fact I think these structures were called *combinatorial manifolds*, hence algebraic topology's deadname "combinatorial topology." Secondly, one should understand that standards for rigor were lower, in part because even conceptual notions in topology lacked systematic language—keep in mind that Frechet, Hausdorff, and others would not establish common notions of a topological space, compactness, metrics, neighborhoods, etc. until 1900-1915. It speaks to Poincaré's great insight and mathematical command, the headway he made so long before anyone reasonably should have.

In 1922, Alexander would investigate a sort of "inside-outside" duality, taking as inspiration the Jordan-Brouwer separation theorem. Recall the 1911 result: a subspace of $\mathbb{R}^n$ homeomorphic to S^{n-1} partitions the plane into two connected components. Alexander investigates the extent to which these components are dual. He elects to projectivize, thus the question is whether $X \hookrightarrow S^n$ and $S^n - X$ are dual, and he ultimately obtains his eponymous duality theorem:

$$\dim H_i(X; \mathbb{Z}/2) = \dim H_{n-i-1}(S^n - X; \mathbb{Z}/2).$$

Again, this is only a result on the level of Betti numbers. The homology of a space was still not exactly defined, and certainly not with coefficients! Alexander was awfully prescient in his analysis, however, and my impression is that this paper paved the way for the definition of relative homology. (I am cautious to say the same for homology, since I assume the idea was already "in the air," although Alexander certainly knew what was going on there too.) I think Lefschetz basically just read Alexander's work and laid down the theory from there (re: relative homology) in his 1930 textbook. Therein, with (co)homology defined absolutely *and* relatively, Lefschetz states a version of Alexander-Lefschetz duality and proves Alexander duality.[16]

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[14] I'm not sure how uniformly true this is.

[15] Moreover, I do not believe it was known when the choice of triangulation mattered. Thus Poincaré's duality was not necessarily intrinsic to a manifold (although maybe folks believed it was), but depended upon the triangulation. Alexander proved invariance under retriangulation over a decade later.

[16] I'm still not entirely confident saying that cohomology is "defined." Lefschetz's book is hard to read. I refer anyone interested to p. 254, VII.8.1 and onward.

Here is a lucid excerpt from Alexander's paper [Ale22] that gives some impression of how the subject was understood at the time. Note Alexander's attention to the infinite nature of his complex / chains within it: I think he was the first to consider homology for infinite complexes.

> If the chain C does not fill up the entire space S^n, the residual part of S^n will form a certain domain $S^n - C$ made up of inner points. We proceed to define the connectivity numbers of this domain. Any chain of any subdivision of S^n will be called a chain of $S^n - C$ provided it is wholly contained in $S^n - C$. Among the chains of $S^n - C$ will be set up the following homologies: (1) Each closed i-chain will be said to be homologous to its derived chains; (2) each closed i-chain which bounds an open $(i + 1)$-chain of $S^n - C$ will be said to be homologous to zero. We combine homologies (1) and (2) like linear equations modulo 2 and denote by $(R^i - 1)$ the maximum number of linearly independent closed i-chains of $S^n - C$. A priori, there is no reason why the number R^i should be finite in this case, since we are now dealing with equations in an infinite number of variables. It will be proved further on, however, that the numbers R^i are all finite and also pure topological invariants of the domain $S^n - C$, in spite of the fact that a metric on S^n has been used in defining them. The number R^0 is of particular importance and evidently denotes the number of separate connected regions in $S^n - C$.
>
> Since we shall only be concerned with the relations between chains under homologies, it will be legitimate to do away with the distinction between a chain of $S^n - C$ and its derived chains. We shall therefore regard any two chains with a common derived chain as equivalent chains, to be denoted by the same symbol K. A closed i-chain will then be said to bound if it bounds in any of its derived forms, so that the terms *bounding* and *homologous to zero* will henceforth be synonymous.

End historical digression. That was made because I was curious when we started thinking about duality between X and $S^n - X$, and was surprised by the answer "from the beginning." Now to return to what I intended to write down, and to speed things up because I have wasted time having fun, let me lay out guiding questions:

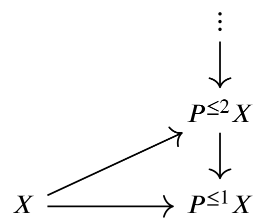
1. Where does Alexander duality live?
2. In what sense is the normal/tangent bundle a stable homotopy invariant?
3. Equivariance?

VI.2 (10/24) Localization III — Slicing

I am reading about the slice filtration in equivariant homotopy theory. Everything equivariant is hard, but I think one can access the slice filtration using rather basic and important ideas (applying hindsight as necessary). So maybe it makes for a good subject for a post. Closely related is some theory regarding localizations.

We can start somewhere elementary. A (pointed) space X is called *k-connective* if $\pi_{i>k}X = 0$. Likewise, we say that X is *k-truncated* if $\pi_{i>k}X = 0$.[17] Given a CW complex X and attaching an $(n+1)$-cell to form X', the (necessarily cofibrant) inclusion $X \to X'$ induces isomorphisms $\pi_{i<n}X \xrightarrow{\sim} \pi_{i<n}X'$ and a surjection $\pi_n X \twoheadrightarrow \pi_n X'$. In particular, we can truncate X.

Definition VI.1 (Postnikov tower)**.** There exists a tower of fibrations



With the following properties.

(I) The space $P^{\leq k}X$ is k-truncated,

(II) The map $X \to P^{\leq k}X$ is a cofibration and an isomorphism on $\pi_{i\leq k}$,

(III) The map $X \to P^{\leq k}X$ is initial among k-truncated maps[18] $X \to Y$,

(IV) The (homotopy) fiber of $P^{\leq k}X \to P^{\leq k-1}X$ is an Eilenberg-Maclane space $K(\pi_k X, k)$.

Good introductory notes for the point-set story are available here (this whole course has great notes). The question of this tower's naturality, uniqueness, functoriality, etc. are highly context-dependent (on e.g. the category at hand, the properties desired). In the context of simplicial sets, wherein spaces are Kan complexes, categories are quasicategories, and functors are coherent, a good construction is possible—in particular, one can exhibit a functorial Postnikov tower of fibrations (in fact we can model it in several ways, although note that the transition maps need not be Kan fibrations for certain Postnikov towers). Probably the best reference for this is Kerodon Section 0513.

The goal today is to place Postnikov truncation in the context of localizations. In particular, I want to explain some theory in the stable setting, and systematize the role of the spheres (i.e., the notion of connectivity). The aim is to understand how to "change our class of spheres" and derive the slice filtration.

The primordial *homotopy groups* spawn a filtration of localizations via the property of *connectivity*. This was already visible in the ordinary homotopy category (c.f. II.2) wherein inverting $i_{n+1} : S^{n+1} \hookrightarrow D^{n+2}$ necessarily inverts all higher i_k. The same occurs without flattening to the homotopy category—e.g. model-categorically, a space X is local with respect to $f = (S^{n+1} \to *)$ if and only if $\Omega^{n+1}X \simeq *$ if and only if X is n-truncated, whence L_f provides a fibrant replacement functor for the model category of spaces whose weak equivalences are the n-connected maps. (In other words, the localization functor whose equivalences only need to be equivalences on the n-truncated homotopy type.) Then we can say that L_f is the n-th Postnikov section $P^{\leq n}$.

We will not proceed model-categorically, but most of the theory was first understood that way. Some good references are [HHR21, Ch. 6] and [Wil17]. The following definition isolates some properties of the subcategory of (co)connected spaces by which it is identified as acyclics for (co)localizations.

[17] This is standard and somewhat displeasing terminology.

[18] I.e., any map $f : X \to Y$ that kills all $\pi_{i>k}X$ factors through $X \to X'$.

Proposition VI.1. A subcategory $\mathsf{C}_0 \subseteq \mathsf{C}$ is called *localizing* if it is closed under extensions and colimits.

Remark VI.1. I think I have found something like six different variations of the above definition.

Remark VI.2. Let C be a stable ∞-category. The (co)connected objects of a t-structure provide important examples of (co)reflective localizations. Not all (co)reflective localizations arise in this manner; such are characterized as being closed under extensions, in which case the acyclic (resp. local) objects constitute the connective (resp. (-1)-coconnective) objects for a t-structure [Lur17, 1.2.1.16]. This is why the extension stability property appears. Noting that $t_{\geq 0}\mathsf{C}$ is a coreflective localization for any t-structure, hence closed under colimits [Lur17, 1.2.1.6], we can say: in the stable setting, a localizing subcategory looks like the connectives for a t-structure ($\iff$ the local objects of a t-colocalization $\iff$ the acyclics of a t-localization).

If C is also presentable, then so is any localizing subcategory and these precisely correspond to connectives for a t-structure c.f. [Lur17, 1.4.4.11]. This is the context we work in.

Example VI.1. The category of n-connective spectra $\mathsf{Sp}_{\geq n}$ is the localizing subcategory generated by $\Sigma^\infty S^n$. The category of $(n-1)$-truncated spectra is the orthogonal complement: we have $X \in \mathsf{Sp}_{\leq n-1}$ if and only if $\mathrm{Map}(Y, X) \simeq 0$ for all $Y \in \mathsf{Sp}_{\geq n}$. This is sometimes written

$$\mathsf{Sp}_{\leq n-1} = (\mathsf{Sp}_{\geq n})^\perp.$$

In other words: the localizing subcategory $\mathsf{Sp}_{\geq n} = \langle \Sigma^\infty S^n \rangle$ occurs as the $t_{\geq 0}$ part of a t-structure such that $t_{\leq -1}$ consists of acyclics against n-connectives, i.e. the $(n-1)$-truncated objects. Since $t_{\geq 0}$ is a colocalization of C, we get a functorial(?) fiber sequence (which could also be obtained in a dual procedure)

$$X_{\geq n} \to X \to X_{\leq n-1}.$$

Here is one special feature of the above example. Each localizing subcategory $\langle \Sigma^\infty S^n \rangle$ gave rise to a t-structure with $t_{\geq 0} = \mathsf{Sp}_{\geq n}$ whose connectives together form the *Whitehead filtration*

$$\cdots \subset \mathsf{Sp}_{\geq 2} \subset \mathsf{Sp}_{\geq 1} \subset \mathsf{Sp}_{\geq 0} \subset \cdots$$

What's special is that this filtration can be generated by a single t-structure: take the standard t-structure (having $t_{\geq 0} = \mathsf{Sp}_{\geq 0}$) and note $\mathsf{Sp}_{\geq 0}[n] = \mathsf{Sp}_{\geq n}$. It is important to note that this is not a generic property of t-filtrations, as we will see.

Fix a finite group G. We next define a new class of spheres $\hat{S}$, and the machinery of localizing subcategories will let us derive an associated filtration of Sp^G by connectivity "relative to $\hat{S}$."

Definition VI.2 ([HHR09])**.** A *slice sphere* is any G-spectrum belonging to

$$\hat{S} := \{G/H_+ \wedge S^{k\rho_H - \epsilon} \,:\, H \leq G \text{ and } \epsilon \in \{0, 1\}\}.$$

Definition VI.3. A G-spectrum is *slice n-connective* if it belongs to the localizing subcategory generated by slice spheres with underlying dimension $\geq n$. We denote this subcategory by

$$\mathrm{P}^{\geq n}\mathsf{Sp}^G := \tau^G_{\geq n}\mathsf{Sp}^G := \mathsf{Sp}^G_{\geq n} := \langle \hat{S} \,:\, \dim \geq n \rangle.$$

Definition VI.4. A G-spectrum Y is *slice n-truncated* if $\mathrm{Map}(X, Y) \simeq *$ for every $X \in \mathsf{Sp}^G_{\geq n+1}$. We denote the full subcategory of these objects by

$$\mathrm{P}_{\leq n}\mathsf{Sp}^G := \tau^G_{\leq n}\mathsf{Sp}^G := \mathsf{Sp}^G_{\leq n} := (\mathsf{Sp}^G_{\geq n+1})^\perp.$$

Example VI.2. Let X be a G-spectrum. If $n \in \{0, -1\}$, then being slice n-(co)connective is equivalent to being n-(co)connective.

Remark VI.3 (c.f. HHR 4.12)**.** The previous statement fails for $n = 1$; e.g. being slice 1-connective does *not* imply the underlying is connected.

Definition VI.5. By the localizing subcategory formalism discussed, we get a localization $P^{\leq n} : \mathsf{Sp}^G \to \mathsf{Sp}^G_{\leq n}$ and colocalization $P_{\geq n} : \mathsf{Sp}^G \to \mathsf{Sp}^G_{\geq n}$ for every n. The inclusions $\mathsf{Sp}^G_{\geq n-1} \subseteq \mathsf{Sp}^G_{\geq n}$ induce natural transformations $P^{\leq n} \to P^{\leq n-1}$, and dually for $P_{\geq n}$.

Definition VI.6. Given $X \in \mathsf{Sp}^G$ we define its *slice tower* as the functorial tower of natural maps $P^{\leq n} X \to P^{\leq n-1} X$ and units for the localizations $P^{\leq n}$. The *n-th slice* of X is the n-th piece

$$P^n_n X := \mathrm{fib}(P^{\leq n} X \to P^{\leq n-1} X).$$

Definition VI.7. The *slice filtration* refers to the sequence of subcategories (or its dual)

$$\cdots \subseteq \mathsf{Sp}^G_{\geq -1} \subseteq \mathsf{Sp}^G_{\geq 0} \subseteq \mathsf{Sp}^G_{\geq 1} \subseteq \cdots$$

Proposition VI.2 (HHR 4.16, 4.45)**.** General localization theory produces two immediate "recognition" lemmas.

1. If $A \to X \to B$ is a fiber sequence such that $A \in \mathsf{Sp}^G_{\geq n}$ and $B \in \mathsf{Sp}^G_{\leq n-1}$, then A, B are isomorphic to $P_{\geq n} X, P^{\leq n-1} X$.
2. (Analogous statement recognizing the slice tower; I think but am not certain that this is an abstract nonsense result, c.f. that one part of HA. Or was it HTT?)

Remark VI.4 (Plug for Dylan Wilson's paper)**.** As in the non-equivariant case, each piece of the slice filtration is induced by a t-structure (as its connectives) by construction. Unlike the non-equivariant case, this filtration need not be generated by a single t-structure: the inclusion $\mathsf{Sp}^G_{\leq 0}[1] = \Sigma \mathsf{Sp}^G_{\leq 0} \subseteq \mathsf{Sp}^G_{\leq 1}$ is usually strict. This means there is a fully faithful functor

$$\mathsf{Sp}^G_{=n} =: \mathrm{Slice}_n \hookrightarrow \heartsuit_n$$

From the n-slice spectra into the heart of the t-structure having $t_{\geq 0}\mathsf{Sp}^G = \mathsf{Sp}^G_{\geq n}$. There is a lot to be said in the direction of describing this functor algebraically:

(I) It is an accessible, left-exact localization.

(II) If $n = k|G|$, then $\heartsuit_n \cong \mathrm{Mack}(G; \mathsf{Ab})$ and the n-slices are those Mackey functors having injective restrictions [HHR09].

(III) For general n, the heart $\heartsuit_n$ may be identified with "modules over a Green functor $\mathrm{End}(W)$ for an isotropic n-slice W." The n-slices occur as those modules with injective restrictions [Wil17, Theorem 2.35].

(IV) For general n, one may identify $\heartsuit_n$ with *twisted Mackey functors*, and the n-slices arise as those satisfying an injectivity condition on their restrictions [Wil17, Theorem 2.82].

(V) You can carry out the above analysis in the more general context of stratified homotopy theories. In this program, the only important property of the set of slice spheres $\hat{S}$ is that they form a poset, carry a weakly increasing dimension function, and "stratify" Sp^G (c.f. [Wil17, Def 1.2]).

This is only an advertisement for Dylan's paper, and we will not discuss the details further.

Now let's get to making some basic results for interacting slice properties with equivariant algebra.

Proposition VI.3 (HHR 4.6-7)**.** Given $H \leq G$, then:

(I) $i^*_H(-)$ sends a G-slice sphere to a wedge of H-slice spheres,

(II) $G_+ \wedge_H (-)$ sends an H-slice sphere to a wedge of G-slice spheres, and

(III) N^G_H sends a wedge of *regular* H-slice sphere to a wedge of regular G-slice spheres.

Proof. (I) and (II) are true by definitions, and (III) is a short application of facts about equivariant indexed algebra. □

Proposition VI.4 (HHR 4.13)**.** Induction and coinduction preserve (co)connectivity degree.

Proof. In the previous proposition, the dimension of underlying spheres is preserved. This implies that coinduction preserves coconnectivity, and that induction preserves connectivity. For the other two statements, apply the Wirthmuller isomorphism and make the same argument. □

Corollary VI.1 (HHR 4.21)**.** $P_G^{\leq n}$ commutes with induction and coinduction.

Generally, things behave better when you narrow your focus to connective objects. This remains true in our situation.

Proposition VI.5 (HHR 4.30-32)**.** In the category of slice connective G-spectra $\mathsf{Sp}^G_{\geq 0}$, the slice n-truncated spectra constitute localizations with localization functor the restriction of $P_G^{\leq n}$. Furthermore, these localizations are weakly monoidal in the sense that $P_G^{\leq n}$

Corollary VI.2. The underlying truncations

Finish.

VII November 2024

VII.1 (11/20) Chromatic I — The Balmer spectrum of Sp

This spring, people seem interested in directing Babytop (our homotopy seminar) toward chromatic. The timing is great: the upcoming Thursday seminar is examining the $L_{K(n)}S^0$ rationalizations paper, and Dhilan and I have vaguely quipped about putting together a semiadditivity seminar, both of which are nicely served by a chromatic primer. Not to mention that learning chromatic is good for my sake. In anticipation, I want to start a series of posts hotfixing my chromatic shortcomings. This is complementary to my goal of understanding "actual" algebraic topology. Some references:

(i) An MO reference request for learning chromatic,

(ii) [Bar+24] "On the rationalizations of the $K(n)$-local sphere" here,

(iii) Barthel and Beaudry's handbook article glimpsing the chromatic philosophy [BB19] here,

(iv) Balmer's handbook article here,

(v) Ravenel's orange book here,

(vi) Sanath's notes here,

(vii) Lurie's notes collated here.

Today I want to collect and organize facts describing the "global chromatic picture" as quickly as possible. I want to stay native to the category of spectra, not yet touching the relation to formal groups. I want to end with the question, "what's up with Morava K-theories?"

It seems economical to begin by introducing a dose of (tensor-)triangulated geometry. The stable homotopy category possesses a *tt*-structure, and to complement the relatively elementary definitions, the *tt*-theory rapidly provides a means to see and discuss the chromatic architecture of (finite) spectra.

Definition VII.1. A *tt-category* is a triangulated category with a biexact symmetric monoidal structure.

Definition VII.2 (Types of subcategories)**.** Let $\mathsf{J} \subset \mathsf{K}$ denote a subcategory of a *tt*-category.

(i) Say J is *triangulated* if J satisfies 2-out-of-3 for exact triangles in K.

(ii) Say J is *thick* if it is triangulated and closed under finite direct sums: $X \oplus Y \in \mathsf{J}$ implies $X, Y \in \mathsf{J}$.

(iii) Say J is a *tt-ideal* if it is thick and closed under tensoring with any object of K.

(iv) Say J is a *prime tt-ideal* if it is a proper *tt*-ideal and $X \otimes Y \in \mathsf{J}$ implies $X \in \mathsf{J}$ or $Y \in \mathsf{J}$.

(v) Say J is *radical* if $X^{\otimes n} \in \mathsf{J}$ for any $n \geq 2$ implies $X \in \mathsf{J}$.

Definition VII.3. The *spectrum of* K is the set $\mathrm{Spc}(\mathsf{K}) := \{\mathsf{P} \subset \mathsf{K} : P \text{ is a prime } tt\text{-ideal}\}$. Given $X \in \mathsf{K}$, its *support* $\mathrm{supp}(X)$ is the subset of prime ideals not containing X.

We regard $\mathrm{Spc}(\mathsf{K})$ as a topological space with a closed basis $\{\mathrm{supp}(X)\}_{X \in \mathsf{K}}$. These constructions mimic those involved in defining the *spectrum of a ring*. The space $\mathrm{Spc}(\mathsf{K})$ has a universal property as the (initial?) space with a support function of closed sets that respects $\oplus, \otimes, \Sigma$ [cf. Balmer].

Theorem VII.1 (Geometric classification of *tt*-ideals)**.**

Corollary VII.1. Given $X, Y \in \mathsf{K}$, we have $Y \in \langle X \rangle$ (= the *tt*-ideal generated by Y) if and only if $\mathrm{supp}(Y) \subseteq \mathrm{supp}(X)$.

We now specialize to the category of spectra. Our thesis is that spectra are a coherently derived enrichment of abelian groups, and there is "prime-by-prime global structure" on $\mathcal{D}(\mathbb{Z})$, thus you should ask about a global structure on spectra lifting it. This is motivational—the details are beside the point, and there are other (more historical) ways to motivate the global chromatic structure. But this is quick and easy.

Let's first say a bit about $\mathcal{D}(\mathbb{Z})$. It has certain distinguished subcategories $\mathcal{D}(\mathbb{Q})$ and $\mathcal{D}(\mathbb{Z})^{\wedge}_p$ together with localization functors to them.[19] The former is the derived category of rational vector spaces; the latter is the category of derived p-complete abelian groups. To quote Barthel-Beaudry, these are "irreducible building blocks" of $\mathcal{D}(\mathbb{Z})$ in the following sense:

(I) The categories $\mathcal{D}(\mathbb{Q})$ and $\mathcal{D}(\mathbb{Z})^{\wedge}_p$ have no nontrivial, proper localizing subcategories.[20]

(II) Any $M \in \mathcal{D}(\mathbb{Z})$ is reassembled as the derived pullback of (p-completions of rationalization + rationalization of p-completions).

Proposition VII.1. This exhausts the minimal, proper localizing subcategories of $\mathcal{D}(\mathbb{Z})$. In other words, we have (the first of the following two) set-theoretic bijections:

$$\mathrm{Spec}(\mathbb{Z}) \cong \{\text{minimal localizing subcategories of } \mathcal{D}(\mathbb{Z})\} \cong \mathrm{Spc}(\mathcal{D}(\mathbb{Z})).$$

Because S^0 is the unit spectrum and $[S^0, S^0] \cong \mathbb{Z}$, the spectrum $\mathrm{Spec}(\mathsf{Sp})$ admits a comparison to $\mathrm{Spec}(\mathbb{Z})$, and the structure on $\mathrm{Spec}(\mathbb{Z})$ described above is refined by structure within $\mathrm{Spec}(\mathsf{Sp})$. Let's explain this.

Remark VII.1 (c.f. [Bal10])**.** The spectrum of a *tt*-category $\mathrm{Spc}(\mathsf{K})$ really is *spectral*: it is homeomorphic to the spectrum of some commutative ring. This follows from Hochster's characterization of spectral spaces, which I do not think is useful for recognizing the ring itself, and this seems beside the point. It is nevertheless worthwhile to ask about an affine parametrization of $\mathrm{Spc}(\mathsf{K})$. One ring presents itself: the endomorphisms of the unit object $\mathrm{End}(\mathbb{1}_{\mathsf{K}})$. (The addition, multiplication, and their compatibility are induced by the additive structure, monoidal structure, and their compatibility, respectively.) Balmer exhibits a natural and continuous map

$$\rho : \mathrm{Spc}\mathsf{K} \to \mathrm{Spec}\mathrm{End}(\mathbb{1}_{\mathsf{K}}).$$

It associates to a prime *tt*-ideal P the endomorphisms f such that $\mathrm{cone}(f) \notin \mathsf{P}$, whch comprise a prime ideal of $\mathrm{End}(\mathbb{1})$ [Bal10, Cor 5.6]. It is generally neither injective nor surjective, but at least surjectivity is a reasonable occurence, and I think one should feel that "the *tt*-spectrum is an enhancement of the endomorphism ring."

Theorem VII.2 (Thm 7.13, [Bal10])**.** If a tensor-triangulated category K is *connected* (i.e. $\mathrm{Hom}_{\mathsf{K}}(\mathbb{1}, \Sigma^i \mathbb{1}) = 0$ for all $i > 0$), then ρ is surjective.

Now consider the following maps:

$$\mathrm{Spc}(\mathcal{D}(\mathbb{Z})) \to \mathrm{Spc}(\mathsf{Sp}) \xrightarrow{\rho} \mathrm{Spec}(\mathbb{Z}).$$

The first map is induced via Spc of the pullback of module spectra along the Hurewicz map $S^0 \to \tau_{\leq 0} S^0 \cong H\mathbb{Z}$. (Here, we need that $\mathsf{Mod}_{H\mathbb{Z}}(\mathsf{Sp}) \cong \mathcal{D}(\mathbb{Z})$.) The second map ρ is the one described above. Noting the previous proposition, the composite is the identity, hence $\mathrm{Spec}(\mathsf{Sp}) \to \mathrm{Spec}(\mathbb{Z})$ is a retract. (We could have also used the previous theorem to conclude it is surjective.) Each fiber of this retract $\mathrm{Spec}(\mathsf{Sp}) \to \mathrm{Spec}(\mathbb{Z})$ admits a sequential filtration by "height," in which sense we get a refinement to Sp of the organization of $\mathbb{Z}$ by residue fields (primes).

(insert picture)

Two guiding comments are in order.

(i) The big mystery now is the nature of the Morava K-theories. For any more than the basic overview of the chromatic picture beyond our sketch above,[21] a serious study of these $K(n)$'s (and related characters) is necessary.

[19] We are talking about (tensor) triangulated categories, in which case I'm not sure what we mean by "localization functors."

[20] A *localizing subcategory* here means a full triangulated subcategory closed under shifts and colimits.

[21] I am not sure if you can even circumvent a somewhat focused study of $K(n)$'s just to get the chromatic filtration—is there a "formal" discernment of the filtration?

(ii) The issue of finiteness is important and recurring. We see it appear already in the structure theorem for the Balmer spectrum of Sp; it is implicit (rather: *omitted* for time's sake) in the problem of classifying *tt*-ideals above; and it will serve a basic role in addressing comment (i). Pay attention to finiteness!

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