2025 Notebook

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I January

I.1 (1/10) Chromatic III – In the year 2025

Christmas and New Year's were short and sweet, and I've just flown back to Cambridge. Continuing my program, I would like to write something about the Morava K-theories. For continuity, let's recap the previous entry.

The derived category of abelian groups decomposes into "irreducible building blocks" in a nice way: it has minimal localizing subcategories $\mathcal{D}(\mathbb{Z})_{\mathbb{Q}}$ and $\mathcal{D}(\mathbb{Z})_p$ for p prime, and in fact any $M \in \mathcal{D}(\mathbb{Z})$ can be reconstructed from its localization data by a certain pullback. These subcategories are examples of *prime ideals in a monoidal triangulated category*, in fact all of them in $\mathcal{D}(\mathbb{Z})$, and the set of them naturally forms a topological space $\mathrm{Spc}(\mathcal{D}(\mathbb{Z}))$. Meanwhile in the category of spectra Sp, the unit S^0 has endomorphism ring $\mathrm{End}(S^0) \cong \mathbb{Z}$ and this begets a natural, continuous map

$$\mathrm{Spc}(\mathrm{Sp}) \to \mathrm{Spec}(\mathbb{Z}) \cong \mathrm{Spc}(\mathcal{D}(\mathbb{Z})).$$

This comparison map is surjective. One can ask about the fibers over each point of $\mathrm{Spec}(\mathbb{Z})$, and the beautiful fact is that each fiber admits a sequential filtration by "height." (A tad more precisely: the fiber consists of a closed point (p), a generic point (0), and a sequence of *intermediary* points which each have closure all the "above" points.)

How can we access this filtration? Let us focus on the finite p-local spectra $\mathrm{Sp}^{\omega}_{(p)}$. It turns out that the global structure of Sp remains interesting upon restriction to $\mathrm{Sp}^{\omega}_{(p)}$, and here we have more proof power. In particular, we can characterize thick subcategories as acyclics for a sequence of awesome spectra $K(p,n)$, and then remove the finiteness assumption. All this and more, coming soon!

(...)

I.2 (1/15) Spectrum of quadratic forms and $\mathbb{S}_{K(1)}$

Recall that $\mathrm{Sp}_{K(n)}$ is ∞-semiadditive, thus every $K(n)$-local spectrum has a unique higher commutative monoid structure. What can be said about that structwure of the unit $\mathbb{S}_{K(n)}$? I have been reading [CY22], whose thesis is that at $p = 2$, the p-typically 1-commutative monoid structure of $\mathbb{S}_{K(1)}$ is captured by a "spectrum of symmetric bilinear forms." More precisely, in [CY22] it is exhibited that

$$L_{K(1)}\mathrm{GW}^{(-)}(\mathbb{F}_\ell) \cong \mathbb{S}_{K(1)}^{(-)}$$

as 1-precommutative monoids, i.e. in the functor category $\mathrm{Fun}(\mathrm{Span}(\mathcal{S}_1^{(p)})^{\mathrm{op}}, \mathrm{Sp}_{K(1)})$. Here, the *Grothendieck-Witt spectrum* $\mathrm{GW}(\mathbb{F}_\ell)$ is the purported "spectrum of symmetric bilinear forms."

Remark I.2.1. To get a whole functor $\mathrm{GW}^{(-)}(\mathbb{F}_\ell)$, we must specify values on BG's and coherences. By construction, at BG this functor obtains a "spectrum of G-equivariant bilinear forms," and the coherences amount to[1] transfers and restrictions.

The spectrum GW is constructed in stages:

(1) Given a symmetric monoidal C, one considers its...

 (i) *Nondegenerate bilinear forms*, understood as the fixed points of the dualizing involution $\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})$. (Understanding Φ is nontrivial because $\mathbb{D}$ lands in C^{op} and thus is not directly instantiated by a C_2-action. This is not the case for the next step.)

 (ii) *Symmetric nondegenerate bilinear forms*, understood as the fixed points $\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})^{hC_2}$ of the C_2-action that precomposes with the swap map.

[1] To what extent do they literally "amount to" this?

(iii) Maximal subgroupoid of the above.

Altogether, we have a space of quadratic forms $QF(\mathsf{C}) := (\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})^{hC_2})^{\simeq}$.

(2) Next, one imagines that C is p-typically m-semiadditive symmetric monoidal. Details must be checked: since m-semiadditivity is preserved by $(-)^{\mathrm{op}}$, we may speak of the category of p-typically m-semiadditive categories with op-involution $(\mathsf{Cat}_m^{(p)})^{hC_2}$ and restrict Φ to it. This category is m-semiadditive and Φ is monoidal, whence $\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})$ is not just an object of $\mathsf{Cat}_\infty^{BC_2}$, in fact it lands in $\mathsf{CMon}_m^{(p)}(\mathsf{Cat}_\infty)^{BC_2}$. Finally, form $QF(\mathsf{C})$ by the same procedure as above. (What's changed is that $QF(\mathsf{C})$ is now a p-typically m-commutative monoid in $\mathcal{S}$.)

(3) There is a group completion functor $(-)^{\mathrm{gp}} : \mathsf{CMon}(\mathcal{S}) \to \mathrm{Sp}$. Applying this "levelwise" (for details c.f. [CY22, §3.2.3]), we compose to a functor

$$GW : \mathsf{CAlg}(\mathsf{Cat}_\infty^{(p),m}) \to \mathsf{CMon}_m^{(p)}(\mathcal{S}) \to \mathrm{PMon}_m^{(p)}(\mathrm{Sp}).$$

The GW functor does not lift to $\mathsf{CMon}_m^{(p)}(\mathrm{Sp})$.[2] However, GW turns out to be lax symmetric monoidal, whence it *does* lift to

$$\mathrm{GW} : \mathsf{CAlg}(\mathsf{Cat}_\infty^{(p),m}) \to \mathsf{CAlg}(\mathrm{PMon}_m^{(p)}(\mathrm{Sp})).$$

Of interest is the following special case (c.f. [CY22, Ex 3.2.13]). For a discete ring R, the category Mod_R is 0-semiadditive; if in addition 2 is invertible in R, then Mod_R is 2-typically 1-semiadditive. We may therefore consider the Grothendieck-Witt theory of R as a 2-typically 1-precommutative monoid (or the quadratic forms $QF(\mathsf{Mod}_R)$ as a 2-typically 1-commutative monoid)

$$\mathrm{GW}^{(-)}(R) : \mathrm{Span}(\mathcal{S}_1^{(2)})^{\mathrm{op}} \to \mathrm{Sp},$$
$$QF^{(-)}(R) : \mathrm{Span}(\mathcal{S}_1^{(2)})^{\mathrm{op}} \to \mathcal{S}.$$

In fact, we can put a CAlg on the target categories, and QF satisfies the Segal condition. (Insert or think about another day: how do we identify with $\mathbb{S}_{K(1)}$ after $K(1)$-localization, and what can we do with that equivalence?)

I.3 (1/20) Chromatic IV – Typically, cooperation is a universal good

(**Abandon hope**: this entry abruptly ends, as I outsourced it to Babytop. Will return later.)

I want to say something about formal groups, Hopf algebroids, and complex cobordism. And more importantly, I want to look inward: what can we say about *ourselves* upon candid, surrendered introspection? Vulnerable reflection is a valuable personal exercise; consider for example the complex cobordism spectrum MU, who opens up upon localization at a fixed prime. Therein we discover remarkable *p-typical* phenomena that power the chromatic machine. To this end, I want to acknowledge the p-typical story and study the height filtration.

Where to start today? As usual, I try to begin with things I know. That would be (commutative, one-dimensional) *formal group laws*. Recall that the *Lazard ring* L is the quotient of $\mathbb{Z}[a_{ij}]$ by the relations necessary for the formal sum $F(x,y) = x + y + \sum a_{ij}x^iy^j$ to be a group law, and by pushing F forward, the Lazard ring L corepresents $R \mapsto \mathrm{FGL}(R)$. Moreover, one may consider L as a graded ring with $|a_{ij}| = 2(i+j)$, and there exists a graded isomorphism $L \cong \mathbb{Z}[x_1, x_2, \dots]$ classifying a group law over $\mathbb{Z}[x_1, \dots]$ which admits an explicit description.

One step to the left, there is a theory of *complex-oriented ring spectra*. One first defines these as ring spectra admitting a theory of Chern classes, with one property relaxed: for such a spectrum E, the first

[2] In order to group complete levelwise, we must first include CMon into PMon, wherein this is possible. This does not rule out the existence of a lift, however the wording of [CY22] suggests they have a counterexample demonstrating nonexistence.

E-Chern class of the tensor product of line bundles is computed as a *formal group law* evaluated on the constituents:

$$c_1^E(\eta \otimes \zeta) = F^E(c_1^E(\eta), c_1^E(\zeta)).$$

(Crucial here is that, somehow, the map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is very cool after hitting it with E^*.) In this way, formal group laws are associated to complex-oriented spectra. The group law lives over E^*. Now, the spectrum MU is tautologically complex-oriented, whence we get a group law F^{MU} over $MU^* \cong \mathbb{Z}[x_1, x_2, \dots]$. Such a thing is classified by a map $L \to MU^*$ and Quillen proved that this is an isomorphism.

We can push this a little further and show that MU is the *universal complex oriented spectrum*. To state this precisely, recall that a *complex orientation* of a commutative ring spectrum E is a chosen class $x^E \in E^*(\mathbb{C}P^\infty)$ satisfying some properties. Something something something [Lur, Lecture 6], [Rog23, Lemma 9.5.2].

finish

Proposition I.3.1.

That concludes the utmost basics. There is ground yet to cover:

(1) A bit more formal group theory—morphisms, strictness, the moduli of formal groups, height.

(2) The stacky perspective and how it informs chromatic—cooperations, Hopf algebras and algebroids, and how to think about the height filtration.

(3) The p-typical story.

In practice, the enumerated topics are intertwined, but it is emotionally easier to sort and separate them. Maybe some goals are to understand the statement "MU_*MU is the universal spectrum with two formal group laws and an isomorphism between them," or "spectra are quasicoherent sheaves on M_{fg}," or "the height filtration on M_{fg} reflects the chromatic filtration." Let's see what exposition I can make here.

Here, maybe we can figure out why cooperations are important, motivate Hopf algebroids, and state the universal property of MU_*MU. This is all maybe most coherently stated in the language of formal groups, but this is sort of *the point*, so if we are to imagine ourselves foreign language students, it's maybe good we do not forego the work of translation.

Cooperations appear because of the Adams spectral sequence. This deserves its own entry, but for now let's be brief. Recall that cohomology operations, or generally transformations $E^*(X) \to D^*(X)$ are induced by homotopy classes of maps $f : E \to D$. The Steenrod squares arise from maps $Sq^i : H\mathbb{F}_p \to \Sigma^i H\mathbb{F}_p$, and these generate the *mod p Steenrod algebra* $\mathcal{A}$ of endomorphisms of $H\mathbb{F}_p$. This is a graded non-commutative algebra with a left action on $H^*(X; \mathbb{F}_p)$, natural in X. One notes that

$$\mathcal{A} \cong H^*(H\mathbb{F}_p; \mathbb{F}_p) \cong [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}.$$

Dually, $H_*(X; \mathbb{F}_p)$ is naturally a left $\mathcal{A}^*$-comodule, and we have $\mathcal{A}^* \cong H_*(H\mathbb{F}_p; \mathbb{F}_p) \cong \pi_{-*}(H\mathbb{F}_p \otimes H\mathbb{F}_p)$. For various reasons, e.g. because $\mathcal{A}^*$ is commutative, it is easier to work with $\mathcal{A}^*$-comodules. For one reason or another, from this structure we are led to the mod p Adams spectral sequence.[3] That's why we care: this helps us compute homotopy groups.

Future entry: substantiate this claim.

(Abrupt ending: I outsourced this project to Babytop 2025.)

I.4 (1/30) Categorical Descent III — Recap

Given objects X and Y, we may be interested in geometric structures $G(X)$ and $G(Y)$ associated to them. Perhaps there is a comparison $G(X) \to G(Y)$, and especially if this map somehow simplifies $G(X)$, we should

[3]Future entry: elaborate.

like to know what data was lost, and whether we can study this data and recover $G(X)$ from it. This is *descent*, which tries to formally equate

$$G(X) = G(Y) + \text{descent data}.$$

The global sections for a sheaf of sets provide a prototypical example: a global section s is the same thing as local sections s_i which agree on overlap. But note there is no descent *data*, as agreement on overlaps is a *property* of local sections. We will come to understand this as a lack of (a choice of) *higher coherences* in a categorical problem.

Before we get on with it, here's an example of descent where we need one more level of coherence: consider a presheaf of *categories* on a space. In fact, let's just consider the whole functor

Fix this discussion. Why did I think this functor was a worthwhile subject? The point is totally obscured.

$$bun : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cat}$$

given by $X \mapsto \mathsf{Top}_{/X}$. We ask if this is a sheaf, and then we wonder what that even means. Let's fix a base space B. AA map $Y \to B$ can be recovered as maps $f_i : Y_i \to U_i$ together with identifications $t_{ij} : f_i^{-1}U_{ij} \xrightarrow{\sim} f_j^{-1}U_{ji}$ satisfying the 1-cocycle condition $t_{jk}t_{ij} = t_{ik}$. I think this amounts to a pullback condition—more precisely, we have an equivalence

$$bun(B) \xrightarrow{\sim} \lim \left(\prod bun(U_i) \rightrightarrows \prod bun(U_i \times_B U_j) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \prod bun(U_i \times_B U_j \times U_k) \right).$$

Encoded on the right-hand side are sections $Y_i \to U_i$ and identifications t_{ij}, with the property of agreement on triple overlaps U_{ijk}. I think this limit must be formed 2-categorically for it to contain sufficient information, c.f. Stacks Project Section 003O. The need for higher categorical limits in order to “correctly" express descent is a recurring problem, and is effectively handled by working with ∞-categories. That is the merit of today's writing, in which I hope to explain how descent is expressed ∞-categorically.

Remark I.4.1. Where I have said “I think," I am truly wary. I am also skeptical of this functor *bun*, which is just the slice over functor. I had only in mind the example where the base B is fixed, and also the covering $\{U_i\}$, I am not certain in what precise sense we can or should say *bun* is a “sheaf." Perhaps it is better to consider the functor given by $X \mapsto \mathrm{Hom}(X, B)$. Or perhaps one should consider a covering $\{U_i\}$ and the map $\coprod U_i \to B$, and consider the induced functor $\mathsf{Top}_{/B} \to \mathsf{Top}_{/\coprod U_i}$.

There are two directions to develop the discussion so far: we can introduce *monads* or we can introduce some *simplicial language* (or both). Today we will make the simplicial language.

We want to understand descent as a sheaf condition. For this, we work in the general setting of a *site*. I will assume the ordinary notion as known; the ∞-categorical notion I will not bother defining for now. Given a cover $\mathcal{U} = \{U_i \to X\}_i$ of an object $X \in \mathsf{C}$, we define its *Cech nerve*

$$N_\bullet(\mathcal{U}) : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{C}$$

by $[n] \mapsto \prod U_{i_1 \dots i_n}$ and (obvious maps; e.g. coface $\delta_i : [n] \to [n-1]$ maps to inclusion into U_J with $i \notin J$). Generalizing the above discussion, we can now take a presheaf $F \in \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{D})$ and ask whether the canonical morphism

$$F(X) \to \lim_{\boldsymbol{\Delta}} F(N_\bullet(\mathcal{U})) \tag{I.4.2}$$

is an isomorphism.[4] If it is, then $F(X)$ determines and is determined by its local sections and gluing data (understood coherently), and we should say that F *satisfies descent for the cover* $\mathcal{U}$. And as descent is the sheaf condition, if F satisfies descent for every cover in the site's topology t, we say *F is a t-sheaf*.

[4]If F does not preserve finite products, then the right-hand side should be interpreted as the limit of the complex with vertices $\prod F(U_I)$, so that the canonical morphism exists.

Example I.4.3. Suppose that the presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ is valued in an n-category.[5] I think that $\mathbf{\Delta}^{\leq n} \hookrightarrow \mathbf{\Delta}$ is n-*final* in the sense that restriction does not change m-categorical limits for $m \geq n$. Take $n = 1$ and $\mathsf{D} = \mathsf{Set}$ or $n = 2$ and $\mathsf{D} = \mathsf{Cat}$, then you recover the sheaf condition for presheaves of sets or categories (or groupoids) discussed above.

Example I.4.4. I think I will consider the case of rings in a separate writing session.

[5] If $n = 1$, this means that D is the nerve of an ordinary category. If $n = 2$, perhaps this means the Duskin nerve? And if $n > 2$, then I only mean this philosophically.

II February

II.1 (2/02) Categorical Descent IV — Faithfully flat descent and the simplicial language

I separately planned to keep with my project of thinking about forms of *descent*, but recently there's been some intersection with my project to learn chromatic. That in mind, I want to present a classical and important example of descent, that being *faithfully flat descent for modules.* There's a homotopical version of this that is sort of important in chromatic land. Let's see what fun we can distill from the ordinary case.

Consider a map of commutative rings $f : R \to S$. The basic question is: given an S-module M, can we find an R-module M^0 and an isomorphism $S \otimes_R M^0 \cong M$? We are asking about filling in the following diagram.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, dashed] & S \arrow[d] \\ M^0 \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & M \end{tikzcd}$$

The data of such a filling, which we should think of as *a descent datum*, is a pair $(M^0 \in \mathsf{Mod}_R, \phi : f^*M^0 \xrightarrow{\sim} M)$. We should like to organize these data into a category.

Here's a fact: given such an (M^0, ϕ), there is a canonical $(S \otimes_R S)$-module isomorphism

$$\tau : S \otimes_R M \cong M \otimes_R S.$$

(This can be deduced a few different ways. It is rather tricky, and I am not sure if there's a canonical reference. I found Jacob Tsimerman's lectures very helpful. Comment from the future: this is Stacks [Sta25, Section 023F].) Moreover, τ satisfies a 1-*cocycle condition* in the sense that the following diagram commutes.

Read the Stacks entry.

$$\begin{tikzcd} & S \otimes_R M^0 \otimes_R S \arrow[dr, "\tau\otimes\mathrm{id}"] & \\ S \otimes_R S \otimes_R M^0 \arrow[ur, "\mathrm{id}\otimes\tau"] \arrow[rr, "\tau'"'] & & M^0 \otimes_R S \otimes_R S \end{tikzcd}$$

We define a *descent datum* as an R-module M^0 together with an isomorphism $S \otimes_R M^0 \xrightarrow{\sim} M^0 \otimes_R S$ satisfying the 1-cocycle condition. These data can be organized into a category $\mathrm{Desc}(f)$. There is an obvious functor $f^* : \mathsf{Mod}_R \to \mathrm{Desc}(f)$, and there is an obvious question: when is f^* an equivalence?

Theorem II.1.1. *If $f : R \to S$ is faithfully flat, then $f^* : \mathsf{Mod}_R \to \mathrm{Desc}(f)$ is an equivalence.*

Proof. See Tsimerman's notes. See also Section 03O6. ☐

Taking Spec, we can translate the above discussion into one about affine schemes: "if $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is faithfully flat and $Z \to \mathrm{Spec}(S)$ is an affine scheme over $\mathrm{Spec}(S)$, then..." We would like to realize this as an equivalence of categories, and for that, we will give another description of $\mathrm{Desc}(f)$. Observe that if f is faithfully flat, descent begets an equalizer diagram

Prove this by hand.

$$M^0 \xrightarrow{\sim} \lim(M^0 \otimes_R S \rightrightarrows M^0 \otimes_R S \otimes_R S).$$

More generally, to $R \to S$ is associated a cosimplicial R-algebra $(S/R)_\bullet := R^{\otimes \bullet+1}$, and to an R-module $R \to M^0$ is associated a cosimplicial $(S/R)_\bullet$-module $(S/R)_\bullet \otimes M^0$. I think it is important here that we are looking at not just a cosimplicial R-module, but a cosimplicial $(S/R)_\bullet$-module (why? I think this is where homotopy theory can enter the picture...)

Example II.1.2 (c.f. Section 023F)**.** One notices that $M^0 \otimes (S/R)_\bullet$ extends the equalizer diagram above. The limit over the full diagram is usually called the *totalization* of $M^0 \otimes (S/R)_\bullet$. Higher descent asks whether the totalization recovers M^0. On the one hand, this is a harder ask, since the totalization diagram is more complicated; on the other hand, since M^0 is "just" a module (as opposed to a cosimplicial one), this might degenerate to the original situation. Well, from a descent datum (M^0, τ) for $f : R \to S$ we can functorially build a degenerate cosimplicial $(S/R)_\bullet$-module $M^0_\bullet$. (How?) This describes a functor

$$\mathrm{Desc}(f) \to \mathrm{Fun}(\Delta, \mathsf{Mod}_{(S/R)_\bullet}).$$

Under this construction, the *canonical* descent datum $(M^0 \otimes_R S, \tau)$ yields $M^0 \otimes (S/R)_\bullet$. Under Dold-Kan, we can turn $M^0_\bullet$ into a cochain complex $s(M^0_\bullet)$. Now we can give an alternate description of descent data, and our intuition on the second hand was correct: if f is faithfully flat, then $\mathsf{Mod}_R \to \mathrm{Desc}(f)$ is an equivalence, with inverse given by $(M^0, \tau) \mapsto H^0(s(M^0_\bullet))$.

In the example, we asked about descent for modules and passed to simplicial junk to factor an inverse-to-be for $\mathsf{Mod}_R \to \mathrm{Desc}(f)$, and concluded that this really did yield an inverse when f is faithfully flat. Now let's do something along these same lines, but starting from $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. We can form a Cech complex

$$Z := (\mathrm{Spec} S/\mathrm{Spec} R)_\bullet := (\mathrm{Spec}(S) \to \mathrm{Spec}(S) \times_{\mathrm{Spec} R} \mathrm{Spec}(S) \rightrightarrows \dots).$$

In the category $\mathrm{Sch}^{\Delta^{\mathrm{op}}}$, there is an obvious factorization $f : \mathrm{Spec}(S) \to Z \to \mathrm{Spec}(R)$. Let $c : Z \to \mathrm{Spec}(R)$ be the latter map. This induces a pullback of quasicoherent sheaves on simplicial schemes $c^* : \mathrm{QCoh}(\mathrm{Spec} R) \to \mathrm{QCoh}(Z)$.

Remark II.1.3. Hold on, wait, what's a (quasicoherent) sheaf on a simplicial scheme $X_\bullet$? We first define a site X_{Zar} whose objects are the opens of the X_n and whose morphisms/covers are the obvious ones (c.f. [Sta25, Section 09VK]). A sheaf on X_{Zar} is equivalent data to a system of sheaves on X_n with compatible cosimplicial maps $X_m \to X_n$ for each $[m] \to [n]$. In particular, we can define the *structure sheaf* $\mathcal{O}_{X_\bullet}$ as that which specifies to $\mathcal{O}_{X_n}$ on every X_n. It is a sheaf of rings on the site X_{Zar}, so we can define its module sheaves and their children (quasicoherent, finite presentation, coherent, etcetera). But here's a hiccup: *the general definition of (say) quasicoherent sheaves does not precisely reproduce "a system of quasicoherent sheaves F_n for each X_n!"* Rather, $\mathrm{QCoh}(X_{Zar})$ is equivalent to *Cartesian* $\mathcal{O}_{X_\bullet}$-modules whose restrictions F_n are quasicoherent $\mathcal{O}_{X_n}$-modules. This is important here, because the "obvious" definition of c^* as pulling back a system of quasicoherent R-modules must be verified to produce cartesian systems. This is also important for another reason, see the final sentence of this section.

Now, I think the maneuver is to interpret $\mathrm{QCoh}(Z)$ as[6] our category of descent data and prove that c^* is an equivalence if f is faithfully flat.

Theorem II.1.4. *If $f : R \to S$ is a faithfully flat map of rings, then*

...

form an equivalence of categories.

I think the interpretation of $\mathrm{QCoh}(Z)$ as descent data is one of the important takeaways here. An important detail is that when you unwind the definition of a quasicoherent sheaf on Z, you get a system of quasicoherent modules $(S^{\otimes n} \to M^0_n)_n$, and this system comprises a descent datum *because it is cartesian* (this is basically the definition of a cartesian sheaf).

Find a reference for this. Check the details.

Remark II.1.5. How to frame this discussion so that the module and scheme discussion can be most concisely compared? In both situations, for instance, we sort of flip-flopped around between ordinary and (co)simplicial stuff...

Remark II.1.6. Recap: notice we did what we intended, that is we reinterpreted *Desc*(f). The goal was to realize it as a category of quasicoherent sheaves, with the secret example of chromatic in mind. This required forming QCoh of a *simplicial* sheaf, though, but miraculously the abstract definition of such a thing produces a system of modules *with the added data (property?)* of being *cartesian*, which is basically what you need for a descent datum.

[6] Or should one imagine $\mathrm{QCoh}(Z)$ as the category of "higher" descent data? Tyler and I convinced ourselves that it is *not* a category of higher descent data, or rather there is no such thing for f, because f is only a map of ordinary rings/affine schemes.

That was a longer entry, and I learned a good deal writing it. Let's reflect.

- I started by defining a descent datum for rings/modules. This is rather unintuitive, only explained by the geometric/Spec story. We already hit everything with Spec later, perhaps best to start with affine schemes? But still want to develop the algebra...
- In general, I think the dual algebra/geometry stories inform one another at different times. If I were to reproduce this discussion for exposition, I may want to be more deliberate in my separation and comparison of the two situations.
- I sort of forgot that $\mathrm{QCoh}(Z)$ was supposed to be our category of descent data, so I should have emphasized that more. Also, I could explain better, and maybe make it more explicit, the way in which $\mathrm{QCoh}(Z)$ encodes descent data.
- I need to do more algebra... Let's maybe wrap up with an exercise.

Here's the basic algebra fact that one might take as a starting point to faithfully flat descent.

Proposition II.1.7. *If $f : A \to B$ is a faithfully flat ring map, then the following is an equalizer diagram.*

$$A \xrightarrow[f]{} B \underset{b\otimes 1}{\overset{1\otimes b}{\rightrightarrows}} B \otimes_A B$$

There are a few ways to rephrase the conclusion that are maybe fun to think about: it asks that $A \to B \times_{B\otimes_A B} B$ be an equivalence, or equivalently that $0 \to A \to B \oplus B \to B \otimes B \to 0$ be exact.

Unwind that last statement?

Proof. We will prove a more general statement, by a long-winded approach touching upon some details overlooked in the above discussion. Let M be an A-module. Recall we defined a cosimplicial B-module $(B/A)_\bullet$ and the $(B/A)_\bullet$-module $(B/A)_\bullet \otimes M$. Formally analogous to the case of sheaves on a site, we may call $(B/A)_\bullet$ the *Cech nerve* of $f : A \to B$, and $(B/A)_\bullet \otimes M$ is the cosimplicial object whose limit we hope to recover as M, in which case we might say $- \otimes_A M$ satisfies descent for f.

Dold-Kan gives a map $s : \mathsf{Mod}_B^\Delta \to \mathrm{CoCh}_{\geq 0}(\mathsf{Mod}_B)$ under which $(B/A)_\bullet$ is given maps $(B/A)_n \to (B/A)_{n+1}$ given by $\sum_i (-1)^i \delta^i_{[n]\to[n+1]}$. Now, consider

$$M \xrightarrow{m\mapsto m\otimes 1} M \otimes B \xrightarrow{m\otimes b\mapsto m\otimes b\otimes 1 - m\otimes 1\otimes b} M \otimes B \otimes B.$$

We will prove that this is exact. In fact, we can prove something even stronger: we recognize this as part of

$$Z := s(M \otimes (B/A)_\bullet) = 0 \to M \to M \otimes B \to M \otimes B \otimes B \to \dots$$

And we will show this cochain complex is exact. Since f is faithfully flat, this is equivalent to the exactness of $Z \otimes B$. Now here's a trick: $Z \otimes B$ is equivalent to $s((M \otimes B) \otimes (B \otimes_A B/B)_\bullet)$! That is to say, there is a natural equivalence

$$s(M \otimes (B/A)_\bullet) \otimes B \cong s((M \otimes B) \otimes (B \otimes B/B)_\bullet).$$

Sounds like a mouthful, and I do not have a reference, but we have reduced to checking the latter is exact, and the latter is the cochain complex obtained from $\Delta_f : B \to B \otimes_A B$ which has a section $b \otimes b' \to bb'$. The point is that we can assume f has a section, and this section allows one to witness every cocycle as a coboundary explicitly, c.f. [Alp24, §2.1.1]. □

Remark II.1.8. What is a good way to think about the role of the diagonal here?

Remark II.1.9. Writing from the future: what I am trying to say in the proof is basically what [Sta25, Section 023F] is trying to say! Let me grab a coffee and summarize this.

We are considering a homomorphism $f : A \to B$ and its induced $f^* : \mathsf{Mod}_A \to \mathrm{Desc}(f)$. We call a descent datum *effective* if it belongs to the image of f^*, and $f^*(M^0)$ is called the *canonical descent datum* associated to M^0. In an attempt to construct an inverse, we consider the functor

$$\mathrm{Desc}(f) \to \mathsf{Mod}^{\Delta}_{(B/A)_\bullet}.$$

The well-definedness of the general definition $(N, \tau) \mapsto N_\bullet$ requires a little work to verify, for which the cocycle condition is crucial. We know what this functor does to canonical descent data: it sends (M^0, can) to $(B/A)_\bullet \otimes M^0$. We also have a functor

$$s : \mathsf{Mod}^{\Delta}_{(B/A)_\bullet} \to \mathrm{CoCh}_{\geq 0}(\mathsf{Mod}_{(B/A)_\bullet})$$

and again, we can describe its effect on (the image of) canonical descent data (M^0, can): we have

$$s((B/A)_\bullet \otimes M^0) = (B \otimes M \to B \otimes B \otimes M \to \cdots).$$

The kernel of the first map $b \otimes m \mapsto b \otimes 1 \otimes m - 1 \otimes b \otimes m$ is M, so we can extend this cochain complex as $0 \to M \to s(...)$. (I want to say that this is an adjoint to one of the good truncations $\mathrm{CoCh} \to \mathrm{CoCh}_{\geq 0}$.) Now, here's a fact.

Proposition II.1.10. *If $f : A \to B$ admits a section, then $0 \to M \to s((B/A)_\bullet \otimes M)$ is exact.*

Proof. Generally, if a morphism $X \to Y$ in C has a section, then the constant cosimplicial object $X_\bullet$ is equivalent to $(Y/X)_\bullet$ assuming the latter can be formed. In our case, we can tensor with M (since functors preserve cosimplicial equivalences) and get $M \simeq (B/A)_\bullet \otimes M$. Now, Dold-Kan is homotopical, so observe that the constant complex on M is exact and $DK((B/A)_\bullet \otimes M)$ is what we wanted to show the exactness of. □

Now here's the trick. If $g : A \to C$ is faithfully flat and the associated complex for $C \to B \otimes_A C$ is exact, then we can conclude $0 \to M \to (B/A)_\bullet \otimes M$ is exact. (This is a definition chase, c.f. [Sta25, Lemma 023M].) The point is that when f is faithfully flat, we have an obvious candidate $g = f$, in which case we're checking $\Delta_f : B \to B \otimes_A B$. The diagonal has a natural section, so the proposition concludes $0 \to M \to s((B/A)_\bullet \otimes M)$ is exact.

The point is that when f is faithfully flat, we know where *effective* descent data goes under the composition $\mathrm{Desc}(f) \to \mathsf{Mod}^{\Delta}_{(B/A)_\bullet} \to \mathrm{CoCh}_{\geq 0}(\mathsf{Mod}_{(B/A)_\bullet})$. Namely, it goes to the complex $s((B/A)_\bullet \otimes M)$ which has $H^0 \cong M$ and trivial higher cohomology. This turns out to characterize the effective descent data:

Proposition II.1.11. *If $f : A \to B$ is faithfully flat, then a descent datum (N, τ) is effective if and only if the canonical map $B \otimes_A H^0 s(N_\bullet) \to N$ is an equivalence.*

Now one can exhibit the equivalence $\mathrm{Desc}(f) \xrightarrow{\sim} \mathsf{Mod}_R$. (...)

II.2 (2/14) Categorical Descent V — fpqc, fppf

> The art of doing mathematics consists in finding that special case which contains all the germs of generality.
>
> David Hilbert

Recall that a quasicoherent sheaf on a scheme X is an $\mathcal{O}_X$-module which is affine locally the (sheaf associated to the) module of global sections; if X is already affine then $\mathsf{Mod}_{\mathcal{O}_X} \cong \mathsf{Mod}_R$. Just as with modules, we can ask about descent for $\mathrm{QCoh}(X)$. The most efficient expression for this is adapted from a fact for modules: a (faithfully) flat ring map $f : A \to B$ is characterized by the (faithful) exactness of $f^* = (-) \otimes_A B$.

We define a *faithfully flat map of schemes* $f : X \to Y$ as one for which $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is faithfully exact. As an exercise, let's unwind this definition a little. We first return to a *ring* map $f : A \to B$ and assume f is flat. We ask what it means for f to be faithful (i.e. for f^* to be faithful, i.e. for f^* to reflect exact sequences).

Proposition II.2.1. *Let $f : R \to S$ be a flat ring map. Then f is faithful if and only if $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proof. [Sta25, Tag 00HP]. □

Likewise, we may suppose that $f : X \to Y$ is flat and ask for a geometric description of its faith. (Flatness is defined Zariski-locally.)

Proposition II.2.2. *Let f be a flat morphism of schemes. Then f^* is faithful if and only if f is surjective.*

Remark II.2.3. Todo: in case of modules, check that reflecting exact sequences implies faithfulness. In case of schemes, find a reference...

Now it turns out that for schemes, faithful flatness is a bit too weak for good descent phenomena. Say $f : X \to Y$ is *fpqc* if it is faithfully flat and every qc open $U' \subset Y$ occurs as $f(U)$ for a qc open $U \subset X$. These include fppf and faithfully flat, quasicompact maps. We can extend faithfully flat descent for modules to fpqc morphisms:

Proposition II.2.4. *Suppose that $f : X \to Y$ is an fpqc map of schemes.*

(1) Given $F, G \in \mathrm{QCoh}(Y)$, the following sequence is exact.

$$\mathrm{Hom}_{\mathcal{O}_Y}(F,G) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}(f^*F, f^*G) \underset{p_1^*}{\overset{p_2^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{X\times_Y X}}(q^*F, q^*G)$$

(2) Suppose as given an $H \in \mathrm{QCoh}(X)$ and an isomorphism $\alpha : p_1^*H \xrightarrow{\sim} p_2^*H$ satisfying $p_{23}^*\alpha \circ p_{12}^*\alpha \cong p_{13}^*\alpha$. Then there exists a unique (M^0, ϕ) where $M^0 \in \mathrm{QCoh}(Y)$ and $\phi : f^*H \xrightarrow{\sim} M^0$ is an isomorphism such that $p_1^*\phi = p_2^*\phi \circ \alpha$.

Proof. This is [Alp24, Prop 2.1.4]. Would be a good idea to come back to this proof. Also maybe §3 in these notes. □

Come back to the proof

Remark II.2.5. It is clear from (2) how to define the category $\mathrm{Desc}(f)$. There is an obvious functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{Desc}(f)$, and conclusions (1) and (2) say that it is fully faithful and essentially surjective, respectively.

In fancier language, the proposition says that QCoh is a stack in the fpqc topology. Later we will make this our standard terminology and make a good theory for stacks; for now, however, more examples. Recall in the topological case my (crude) attempt to explain how the functor $X \mapsto \mathsf{Top}_{/X}$ demonstrates stack-like properties. Well, here's something that looks similar, at least with respect to fpqc morphisms.

Proposition II.2.6. *Let $f : X \to Y$ be an fpqc morphism of schemes. For any morphism $g : X \to Z$ equalized by the projections $p_1, p_2 : X \times_Y X \to Y$, there exists a unique h such that the following diagram commutes.*

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & & Z \end{tikzcd}$$

This translates to $\mathrm{Hom}(-, Z)$ being a stack in the fpqc topology. There is an analogous statement in the relative situation for $\mathrm{Sch}_{/S}$. (Later, we say these are statements about stackiness over the "big" and "small (over S)" fpqc site.)

How to compare to prestack sending scheme Z to morphisms of *algebraic spaces* to Z? This is "really" the stack I was alluding to earlier.

A morphism $f : X \to Y$ is called *fppf* if it is faithfully flat and locally finitely-presented. These morphisms are fpqc, hence the previous descent phenomena apply, however one can say more.

II.3 (2/26) Chromatic V — Formal groups

Two Babytalks have passed. Yesterday, Oakley reviewed "MU-theory," defined formal groups, and suggested the perspective that spectra (via their MU-homology) are quasicoherent sheaves on $\mathcal{M}_{\mathrm{fg}}$. I have neglected my chromatic notebooking in favor of organizing Babytop, first to avoid wasteful overlap, and second, to learn more before writing things down. Now that we have covered some ground (and also *not* covered some ground), I'm more confident I can say something useful and productive. Today I want to think about *formal groups*.

- Piotr's notes [Pst] are a great reference. One goal of the seminar is to get everyone comfortable with the language of these notes (or, say, Lurie's notes, or Goerss's notes).

The natural context for a formal group is the category of *formal schemes*. These are not exactly ind-schemes, although there's a relationship that's yet unclear to me—but in any case, we care about the local geometry of our formal schemes, so a study from the ground up is quite necessary.

Functor-of-points compels you to identify a commutative ring $A \in \mathsf{CRing}$ with its presheaf $\mathrm{Spec}(A) := \mathrm{Hom}_{\mathsf{CRing}}(A, -) \in \mathrm{Fun}(\mathsf{CRing}, \mathsf{Set})$. An ideal $I \subset A$ has an associated *I-adic completion* $\hat{A}_I := \lim(A/I \leftarrow A/I^2 \leftarrow \cdots)$, and dually we define the associated *formal completion* of $X := \mathrm{Spec}(A)$ along the closed subscheme $Z := \mathrm{Spec}(A/I)$ as the colimit

$$\hat{X}_Z(-) = \operatorname{colim} \mathrm{Hom}_{\mathsf{CRing}}(A/I^n, -).$$

In words, $\hat{X}_Z$ is the subfunctor of $\mathrm{Spec}(A)$ which associates to B those morphisms $A \to B$ annihilating some I^n (hence vanishing near zero). A functor $\mathsf{CRing} \to \mathsf{Set}$ of this form is called an *affine formal scheme*.

Remark II.3.1. The filtration $A \supseteq I \supseteq I^2 \supseteq \cdots$ induces an cofiltered/inverse/projective system $A/I \leftarrow A/I^2 \leftarrow \cdots$. The *limit topology* on $\hat{A}_I$ is the initial having every $\phi_n : \hat{A}_I \to A/I^n$ continuous with respect to the discrete topology on A/I^n. It is linearized by the kernels $\ker(\phi_i)$, i.e. it is the associated filtration topology. The canonical $A^{adic} \to \hat{A}_I$ is continuous, in fact it is the initial map to a *separated, Cauchy complete, linearly topologized* ring [Sin11, 8.2.4]. This includes any ring with the discrete topology; moreover observe the (0)-adic topology is discrete, so you make this a statement about Hom-sets of (complete adic rings? But what about I not finitely-generated?)

Definition II.3.2. Consider A with an I-adic topology. A continuous ring map $A^{adic} \to B^{disc}$ is a ring map $A \to B$ such that some I^n is annihilated, whence we can identify

$$\operatorname{colim} \mathrm{Hom}_{\mathsf{CRing}}(A/I^n, B) \cong \mathrm{Hom}^{cts}_{\mathsf{CRing}}(A^{adic}, B^{disc}) \cong \mathrm{Hom}_{\mathsf{CRing}}(\hat{A}_I, B).$$

Thus, the affine formal scheme arising from (A, I) depends only on the topology A^{adic} and not the specific ideal of definition I. The resulting sheaf is often called the *formal spectrum* $\mathrm{Spf}(A)$.

Example II.3.3. The *formal affine line* over R is the formal spectrum $\hat{\mathbb{A}}^1_R := \mathrm{Spf}(R[\![t]\!])$ where $R[\![t]\!]$ carries the t-adic topology. In effect, $\mathrm{Spf}(R)(S)$ consists of pairs $(R \to S, x)$ of R-algebras and a choice of nilpotent element. In particular, $\hat{\mathbb{A}}^1(S) := \mathrm{Spf}(\mathbb{Z}[\![t]\!])(S) = \mathrm{Nil}(S)$.

Definition II.3.4. Let F be a formal group law over R. Its associated *formal group* is the functor

$$G_F : \mathrm{Alg}_R \to \mathsf{Ab} \quad \text{given by} \quad S \mapsto (\mathrm{Nil}(S), x + y := F(x, y)).$$

Thus, the formal group G_F is the lift of $\mathsf{CRing}_{R/} \xrightarrow{\hat{\mathbb{A}}^1_R} \mathsf{Set}$ to Ab defined by designating F as the group operation. We can ask about the set of such lifts, i.e. the set of abelian group structures on $\hat{\mathbb{A}}^1_R$. Before specifying to studying maps $(\hat{\mathbb{A}}^1_R)^2 \to \hat{\mathbb{A}}^1_R$, we may consider maps from an (almost) arbitrary $\mathrm{Spf}(A)$.

Proposition II.3.5. *If an R-algebra A is I-adically complete, then* $\mathrm{Hom}_{/\mathrm{Spec}(R)}(\mathrm{Spf}(A), \hat{\mathbb{A}}^1_R) \cong \mathrm{Nil}^{top}(A)$.

Proof. Acknowledge the monomorphism $\hat{\mathbb{A}}^1_R \hookrightarrow \mathbb{A}^1_R$, and we can now first figure

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spf}(A), \mathbb{A}^1_R) \cong \lim \mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spec}(A/I^n), \mathbb{A}^1_R) \cong \lim \mathrm{Hom}_{\mathrm{Alg}_R}(R[t], A/I^n) \cong A.$$

The last equivalence is due to completeness. Now we ask which $a \in A$ factor through $\hat{\mathbb{A}}^1_R \hookrightarrow \mathbb{A}^1_R$, and by the magic of monomorphisms this just means computing

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spf}(A), \hat{\mathbb{A}}^1_R) \cong \lim \mathrm{Hom}(\mathrm{Spec} A/I^n, \hat{\mathbb{A}}^1_R) \cong \lim \mathrm{Hom}^{cts}(R[\![t]\!], A/I^n).$$

I think the RHS just unwinds to $\mathrm{Nil}^{top}(A)$. Wait, could I have just done that to start? $\square$

Unwind and check this...

Corollary II.3.6. *There is a bijection*

$$\mathrm{Hom}_{\acute{e}t,/\mathrm{Spec}(R)}(\hat{\mathbb{A}}^n_R, \hat{\mathbb{A}}^1_R) \cong \{F \in R[\![x_1, \ldots, x_n]\!] : F \textit{ has nilpotent constant term}\}.$$

It follows that abelian monoid structures (automatically grouplike) on $\hat{\mathbb{A}}^1_R$ for which zero is a unit are in bijection with formal group laws over R. This makes the following definition a bit easier to digest.

Definition II.3.7. A *formal group* over $\mathrm{Spec}(R)$ is an abelian group object

$$\mathbf{G} \in \mathrm{Shv}_{\text{ét}}(\mathsf{CRing}_{R/}, \mathsf{Ab})$$

such that Zariski locally, $\mathbf{G}$ takes the form of a formal group associated to some formal group law.

Remark II.3.8. What a word salad. Let me unwind this. Before all else, $\mathbf{G}$ is an étale sheaf $\mathrm{Alg}_R \to \mathsf{Ab}$. Moreover, there exists a finite list of elements $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and for each restriction $\mathbf{G}|_i : \mathrm{Alg}_{R_{f_i}} \to \mathsf{Ab}$, there exists an isomorphism $\mathbf{G}|_i \cong G_F$ for some F.

Example II.3.9. The most important example: if E is complex-orientable, then $\mathrm{Spf}(E^*\mathbb{C}P^\infty)$ is a formal group over E^*. To explain, we will assume $E^* = \pi_* E$ is even, which avoids some unfathomable problems. By orientability, there exists an isomorphism $E^*\mathbb{C}P^\infty \cong \lim E^*\mathbb{C}P^n$ by which we can give $E^*\mathbb{C}P^\infty$ the limit topology. We may well enough form

$$\mathrm{Alg}_{E^*} \xrightarrow{\mathrm{Spf}(E^*\mathbb{C}P^\infty)} \mathsf{Ab}$$

and try to realize $\mathrm{Spf}(E^*\mathbb{C}P^\infty)$ as some G_F. A choice of orientation $t \in E^2(\mathbb{C}P^1)$ determines an isomorphism $E^*\mathbb{C}P^\infty \cong E^*[\![t]\!]$, and this isomorphism is t-adically continuous. We may write $\mathrm{Spf}(E^*\mathbb{C}P^\infty)(S) = \mathrm{Hom}^{cts}_{E^*}(E^*[\![t]\!], S) = \mathrm{Nil}(S)$. The group structure on this hom-set comes from that on $E^*[\![t]\!]$, hence from $E^*\mathbb{C}P^\infty$, hence from the tensor product of line bundles, which is the formal group law we know and love.

In the next entry, I want to collab with my Categorical Descent musings to form the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups, and explain how the geometry of $\mathcal{M}_{\mathrm{fg}}$ informs our understanding of spectra.

III March

III.1 (3/1) The Picard group of a category

Given a (commutative) monoid M, one may form its *(commutative) group of units* $M^\times$, and there is an obvious functor $M \mapsto M^\times$ which is *right* adjoint to the inclusion $\mathsf{Gp} \hookrightarrow \mathsf{Mon}$. The homotopy-coherent story asks about a right adjoint to the inclusion

$$\mathrm{Sp}_{\geq 0} \hookrightarrow \mathsf{CMon}(\mathcal{S}).$$

III.2 (3/4) Descent & Chromatic VI — A map $\mathcal{M}_{MU} : \mathrm{Sp} \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$

III.3 (3/7) Spectral Sequences I

An often sufficient (but not necessary) way to recognize an algebraic topologist is look for an ability to maneuver spectral sequences. I have said before that my formal topology training was not well-formed, and I'm unfamiliar with spectral sequences as a result. Andy is giving some number of lectures introducing spectral sequences, and he's the good stuff, so I will be writing some things down related to what he says.

Here's sort of the basic germ of the way that (Serre) spectral sequences arise in algebraic topology. Consider the Hopf fibration

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2.$$

Fibrations induce long exact sequences in homotopy. What can be said about homology? The answer is more sophisticated than a long exact sequence. Homology is built out of *simplices*, so we wonder how a fibration induces some disfigurement of simplices. Fix a relative homeomorphism $\sigma : (\Delta^2, \partial\Delta^2) \xrightarrow{\sim} (S^2, *)$, which determines a generating cycle $[\sigma] \in H_2(S^2)$.[7] Since $\Delta^2 \cong I^2$, the lifting property gives us a filling

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & S^3 \arrow[d, "p"] \\
(\Delta^2, \partial\Delta^2) \arrow[r, "\sigma"'] \arrow[ur, dashed, "\bar{\sigma}"] & S^2
\end{tikzcd}$$

The resulting 2-simplex $\bar{\sigma}$ is *not* a cycle—this would require its boundary be trivial, but the lift has an "error" term: it need only map $\partial\Delta^2$ to a fiber circle S^1, in fact[8] it must wind nontrivially around S^1, in fact it must *generate* $H_1(S^1)$. In this sense, the generator of $H_2(S^2)$ failed to lift to $H_2(S^3)$ because it was obstructed by the nontriviality of its boundary in $H_1(S^1)$. We've learned some things:

(i) We realized the generator of $H_1(S^1)$ as a boundary in S^3, so $H_1(S^1) \to H_1(S^3)$ has trivial image.

(ii) It is impossible to lift the generator of $H_2(S^2)$ to $H_2(S^3)$, so $H_2(S^3) \to H_2(S^2)$ has trivial image.

One deduces that $H_1(S^3)$ and $H_2(S^3)$ are trivial. (How exactly does this follow?) One should view the described map $d : H_2(S^2) \to H_1(S^1)$ as controlling the situation, in particular its nontriviality has consequences. This is our first example of a *differential*.

Remark III.3.1. If we instead consider the trivial bundle $S^1 \times S^2 \to S^2$, one could compute $H_*(S^1 \times S^2)$ via Künneth; this fails for the Hopf fibration, as Künneth predicts "too many" elements of $H_1(S^3)$and $H_2(S^3)$, coming from $H_1(S^1)$ and $H_2(S^2)$, respectively. Above we described a nontrivial map $d : H_2(S^2) \to H_1(S^1)$ induced by the Hopf fibration (an isomorphism); the analogous map induced by $S^1 \times S^2 \to S^2$ is trivial. These facts are not coincidental: the map d is our first example of a *differential*, whose nontriviality is strongly related to Künneth's misprediction.

[7] Maybe you need $[\sigma - *]$?

[8] Why is this true? Does one use the nontrivial fiber bundle structure? How to phrase this using only the fact that p is a fibration?

Above, our answer to "what can we say about homology given a fibration?" was answered in the example of the Hopf fibration. We used the HLP to extract an algebraic map d which told us things. This is a simple example of an algebraic structure which is generally complicated, which we define now. The generic machinery is derived from *filtrations*, and we will specialize back to the case of spaces and fibrations via skeletal filtrations.

We start by considering a chain complex $C_\bullet$ and a *filtration* by subcomplexes

$$F_0C_\bullet \subseteq F_1C_\bullet \subseteq \cdots$$

Note that the boundary for $C_\bullet$ moves orthogonal to the filtration, i.e. it has bidegree $(0, -1)$. For example, if $x \in F_sC_5$, then $dx \in F_sC_4$. If we are interested in the homology of $C_\bullet$, then we are interested in dx, and our filtration coordinatizes an approximation of dx. This is the slogan which spectral sequences formalize. The first step in describing this approximation is to form the "leading term" of dx as its quotient $[dx]_s \in F_sC_4/F_{s-1}C_4$.

If $[dx]_s = 0$, then we conclude $dx \in F_{s-1}C_4$ and repeat for $[dx]_{s-1} \in F_{s-1}/F_{s-2}$. To formalize this, we form the *associated graded* pieces

$$0 \to F_{s-1}C_\bullet \hookrightarrow F_sC_\bullet \twoheadrightarrow \mathrm{gr}_sC_\bullet \to 0 \quad \rightsquigarrow \quad \mathrm{gr}_\bullet C := \bigoplus_i \mathrm{gr}_i C.$$

The associated graded $\mathrm{gr}_\bullet C$ is a graded object in chain complexes. In our example of $x \in F_sC_5$, we sought to understand $dx \in F_sC_4$, and we proposed to interpret $[dx]_s \in \mathrm{gr}_sC_4$ as the "leading term in an approximation of dx determined by the filtration." In the general setup, we will use the homologies of the complexes in $\mathrm{gr}_\bullet C$ to track these approximations.

Definition III.3.2. We form the bigraded abelian group $E^0_{s,t} := \mathrm{gr}_sC_{s+t} = F_sC_{s+t}/F_{s-1}C_{s+t}$. The boundary from C induces a bidegree $(0, -1)$ differential $d^0\colon E^0_{s,t} \to E^0_{s,t-1}$.

Definition III.3.3. The homology of the bicomplex $(E^0_{\bullet,\bullet}, d^0)$ is a bigraded abelian group which we denote

$$H_{s,t}(E^0_{\bullet,\bullet}) =: E^1_{s,t}.$$

Similar to $\mathrm{gr}_\bullet C$, the $E^1_{s,t}$ form a bigraded abelian group.

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