2026 Notebook

Matthew Niemiro

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I.1 (1/5) Integration

I've needed to think about integration recently. It has been a while since I last got into the details, and I need to do some reflection to prepare my reading seminar's syllabus anyway, so I'll write a bit here. Today I just want to unwind several instances of semiadditivity and the accompanying integration.

Example I.1.1 (Ordinary integration)**.** Consider the ordinary category of abelian groups Ab. It has a zero object and admits finite biproducts, hence it is *zero semiadditive*. This induces a canonical enrichment in commutative monoids, which we may understand as a type of integration: for any finite family of maps $f : S \to \mathrm{Map}_{\mathsf{Ab}}(A, B)$ we may form the sum

$$(\int_S f)(a) := \sum_S f_s(a).$$

We can describe this categorically. First, identify f as a single morphism $\overline{f} : \coprod_S A \to B$. We would like to define $\int_S f$ by postcomposing this with a "diagonal into the coproduct" $A \to \coprod_S A$, which does not exist *a priori*. What do exist *a priori* are the diagonal $\Delta : A \to \prod_S A$ and the canonical comparison $\mathrm{Nm}_S : \coprod_S A \to \prod_S A$. Fortunately, the semiadditivity of Ab means that Nm_S is invertible, and we may form

$$\int_S f \quad \text{as the morphism} \quad A \xrightarrow{\Delta} \prod_S A \xrightarrow{\mathrm{Nm}_S^{-1}} \coprod_S A \xrightarrow{\overline{f}} B.$$

We can phrase this ability to integrate $f : S \to \mathrm{Hom}_{\mathsf{Ab}}(A, B)$ more functorially and in a manner more intrinsic to the set S. For this, consider the function $q : S \to *$ and the pullback $q^* : \mathsf{Ab} \to \mathsf{Ab}^S$. Its adjoints $q_!$ and q_* send S-families to their (co)products, and the canonical comparison is a transformation $\mathrm{Nm}_q : q_! \to q_*$. What are S-families of morphisms? We can identify (by mating along the expoenntial adjunction) $\mathrm{Hom}_{\mathsf{Ab}^S}(q^*X, q^*Y) \cong \prod_S \mathrm{Hom}_{\mathsf{Ab}}(X, Y)$, hence we may regard f as a morphism $q^*X \to q^*Y$. Then we can also mate f along $q_! \dashv q^*$ to $\overline{f} : \coprod_S X = q_!q^*X \to Y$. That said, everything necessary to integrate the family f has now been defined in terms of q. We define *integration* as the function

$$\begin{array}{rccccc} \int : & \mathrm{Map}(q^*A, q^*B) & \longrightarrow & \mathrm{Map}(q_!q^*A, B) & \longrightarrow & \mathrm{Map}(A, B). \\ & f & \longmapsto & \overline{f} & \longmapsto & \overline{f} \circ \mathrm{Nm}_q^{-1} \circ \Delta_q \end{array}$$

One upshot of this approach is that it becomes manifest that $\int$ is functorial in A and B. It also is the "correct" approach to generalize, which is why I've written out this example here. Here's a little "cheat sheet" for the relative language.

Absolute notation	**Relative notation for** $q : S \to *$
$\coprod_S X, \prod_S X$	$q_!q^*X, q_*q^*X$
$\Delta_S : X \to \prod_S X$	$u_* : X \to q_*q^*X$
$f : S \to \mathrm{Map}_C(X, Y)$	$f : q^*X \to q^*Y$
$\overline{f} : \coprod_S X \to Y$	$\overline{f} : q_!q^*X \xrightarrow{f} q_!q^*Y \xrightarrow{c_!} Y$

Example I.1.2 (Absolute integration given m-semiadditivity)**.** Consider an m-semiadditive ∞-category C. This grants, for every m-finite diagram $F : A \to C$, an *inverse* to the norm map $\mathrm{Nm}_F^{-1} : \lim_A F \xrightarrow{\sim} \mathrm{colim}_A F$. This lets us integrate a family $f : A \to \mathrm{Map}_C(X, Y)$ by the same procedure as above. Namely, we may form

$$\int_A f \quad \text{as the morphism} \quad A \xrightarrow{\Delta_A} \operatorname*{colim}_A X \xrightarrow{\mathrm{Nm}_A^{-1}} \lim_A X \xrightarrow{f} Y.$$

Also as before, we can phrase this categorically, and this will have the advantage of generality and naturality. (Also as before, this will emphasize the role of the space A, or rather the map q.) For this, consider the map $q : A \to *$. Then we may once again define

$$\int_A : \mathrm{Map}_{C^A}(q^* X, q^* Y) \to \mathrm{Map}_C(X, Y),$$

which now occurs as a map of anima.

Example I.1.3 (Relative integration)**.** In fact, m-semiadditivity buys a little more. For an arbitrary map of m-finite spaces $q : A \to B$, the pullback q^* admits adjoints $q_!, q_*$ because C admits $(m-1)$-finite (co)limits, a canonical norm $\mathrm{Nm}_q : q_! \to q_*$ because C admits $(m-1)$-finite bilimits, and an inverse $\mathrm{Nm}_q^{-1} : q_* \xrightarrow{\sim} q_!$ because C is m-semiadditive. (The extra generality is that $B \neq *$.) From here, we can perform the same procedure as above to define integration. As we are handling ∞-categories, we should like to be careful and explicit about how we do this. I'll follow the formalism of [CSY22]: we define integation as the composite

$$\begin{array}{lccccccc} \int_q : & \mathrm{Map}_{C^A}(q^* X, q^* Y) & \xrightarrow{q_!} & \mathrm{Map}_{C^B}(q_! q^* X, q_! q^* Y) & \xrightarrow{c_{!,Y} \circ -} & \mathrm{Map}_{C^B}(q_! q^* X, Y) & \xrightarrow{- \circ \mathrm{Nm}_q^{-1} \circ u_*} & \mathrm{Map}_{C^B}(X, Y) \\ & f & \longmapsto & q_! f & \longmapsto & \overline{f} & \longmapsto & \overline{f} \circ \mathrm{Nm}_q^{-1} \circ u_{*,X} \end{array}$$

This has equivalent formulations which I won't mention, c.f. [CSY22, Remark 2.12]. It may be enlightening to note that the unit $u_* : \mathrm{id} \to q_* q^*$ is the q-diagonal, that $c_{!,Y} \circ q_! f$ is the adjunct $\overline{f}$, and that $\mathrm{Nm}_{q,q^* X}^{-1} \circ u_{*,X} : X \to q_* q^* X \to q_! q^* X$ is the *wrong-way unit* which is closely related to the ambidexterity of q [CSY22, 2.17-8].

Remark I.1.4 (Norms via integration). The slogan is that m-semiadditivity permits the functorial integration of m-finite families of morphisms. This is a key feature of semiadditivity (and ambidexterity). In fact, integration is how you construct norm maps. For this, suppose that C is m-semiadditive and that $F : K \to C$ is an $(m+1)$-finite diagram. We would like to exhibit a morphism

$$\mathrm{Nm}_F : \operatorname*{colim}_K F \to \lim_K F.$$

Of course, we must suppose C admits such (co)limits. Then informally, such a morphism Nm_q should look like a compatible family of morphisms

$$\varphi_{k_1,k_2} : F(k_1) \to F(k_2)$$

for every pair of points $k_1, k_2 \in K$. There is *a priori* no canonical such map, but there *is* a canonical family of such maps $F_{k_1,k_2} : \mathrm{Map}_K(k_1, k_2) \to \mathrm{Map}_C(F(k_1), F(k_2))$. Because K is $(m+1)$-finite, the path-space $A := \mathrm{Map}_K(k_1, k_2)$ is m-finite. The path forward is now clear: tranpose F_{k_1,k_2} along the exponential adjunction to $\overline{F}_{k_1,k_2} \in \mathrm{Map}_{C^A}(q^* F(k_1), q^* F(k_2))$ and define

$$\varphi_{k_1,k_2} := \int_A \overline{F}_{k_1,k_2}.$$

The assemblage of these define Nm_F: $\operatorname{colim} F \to \lim F$.

I.2 (1/10) Parametrized integration

The parametrized theory replaces the role of spaces with an arbitrary ∞-category $\mathcal{X}$ with pullbacks and a terminal object (or something along those lines). Such parametrized categories admit notions of (co)completeness, adjoints, norms, and m-semiadditivity. In particular, they should admit a notion of integration. I would like to explain this without getting too much into the weeds, if that's possible.

A parametrized category is simply a functor $C : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$. We assume $\mathcal{X}$ has a terminal object, thus each object X has an accompanying map $t_X : X \to *$.[1] This induces a functor $t_X^* : C_{\mathrm{pt}} \to C_X$. If C is $\mathsf{S}_{\pi\text{-fin}}$-(co)complete, then t_X^* admits left and right adjoints.

Definition I.2.1. Suppose that $C : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ is parametrized (co)complete. Then for $X \in \mathcal{X}$ and $M \in C_{\mathrm{pt}}$, we associate to M its

$$\begin{aligned} (t_X)_! t_X^*(M) : C_{\mathrm{pt}} \to C_{\mathrm{pt}} & \quad \textit{parametrized homology} \text{ and} \\ (t_X)_* t_X^*(M) : C_{\mathrm{pt}} \to C_{\mathrm{pt}} & \quad \textit{parametrized cohomology}. \end{aligned}$$

For $M \in C_{\mathrm{pt}}$, we may write $M_C[X]$ and M_C^X for its homology and cohomology, respectively.

Example I.2.2. Associated to Sp is the S-category $\mathrm{Sp} : \mathsf{S}^{\mathrm{op}} \to \mathsf{Cat}$ given by $X \mapsto \mathrm{Sp}^X$. (You can generally parametrize an ∞-category in this manner, and certain categorical properties can be recharacterized, c.f. for instance [Ben24, §2.2].) Here, for a space X and spectrum E, we have

$$\begin{aligned} E_{\mathrm{Sp}}[X] &= (t_X)_! t_X^* E = \operatorname*{colim}_X E = E \otimes X = E \wedge \Sigma^\infty_+ X, \quad \text{and} \\ E_{\mathrm{Sp}}^X &= (t_X)_* t_X^* E = \lim_X E = E^X = \mathrm{Map}(\Sigma^\infty_+ X, E), \end{aligned}$$

as the notation would suggest.

This entry is supposed to say something about *parametrized integration*. It may not yet be clear how we're getting there. The idea is to categorify the properties of (co)completeness, semiadditivity, span constructions, etc. to the parametrized setting, then realize (co)homology as a form of decategorification, then use the identification of homology and cohomology (in the presence of semiadditivity) to define integration as the induced wrong-way map on cohomology

$$\int : M_C^X \cong M_C[X] \to M_C[Y] \cong M_C^Y.$$

Now, let's do that analysis. We will rely heavily on the relationship between semiadditivity and spans. Recall that relation: spans are a categorical syntax for coherent, simultaneous covariant and contravariant functoriality, and consequently they are an intrinsic language for the mechanics of semiadditivity, especially if one is interested in functoriality.

Definition I.2.3. Let $\mathcal{X}$ denote a category with pullbacks, and write $\mathrm{Span}_{1.5}(\mathcal{X})$ for the $(\infty, 2)$-category of spans. There is a canonical functor

$$\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_{1.5}(\mathcal{X}).$$

Under this map, a morphism $f : X \to Y$ in $\mathcal{X}$ is sent to $f^* = \left(Y \xleftarrow{f} X = X\right)$. There is also an associated morphism $f_! = \left(X = X \xrightarrow{f} Y\right)$ which fits into an adjunction $f_! \dashv f^*$ within $\mathrm{Span}_{1.5}(\mathcal{X})$ with $u_! = \Delta_f$ and $c_! = f$.

For a 2-category $\mathbf{A}$, a parametrized $\mathbf{A}$-object $F : \mathcal{X}^{\mathrm{op}} \to \mathbf{A}$ is called *$\mathcal{X}$-cocomplete* if for every $f \in \mathcal{X}$, the morphism $f^* = F(f)$ admits a left adjoint $f_!$, and furthermore the $BC_!$ condition holds for pullbacks. This first condition is satisfied for $\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_{1.5}(\mathcal{X})$. In fact, it also satisfies the second condition, in a universal way.

Theorem I.2.1 ([Ben24], [Mac22])**.** *Let $\mathcal{X}$ denote a category with pullbacks. Then the map*

$$\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_{1.5}(\mathcal{X})$$

[1] People like to call this f_X or q_X, but I like to use t for terminator, which is less overloaded.

exhibits $\mathrm{Span}_{1.5}(\mathcal{X})$ *as the universal 2-category with an* $\mathcal{X}$*-cocomplete object. In other words, for a 2-category* $\mathbf{A}$*, pre-composition induces an isomorphism*

$$\mathrm{Map}(\mathrm{Span}_{1.5}(\mathcal{X}), \mathbf{A}) \xrightarrow{\sim} \mathrm{Map}^{cc}(\mathcal{X}^{\mathrm{op}}, \mathbf{A}).$$

Remark I.2.4. This result is cool.

Elaborate

By some sequence of events which I don't think could exist peacefully in this entry, one can also understand a related universal property of the slice functor as a parametrized category $\mathcal{X}_{/-} : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$.

Future entry: parametrizing in $\mathbf{A}$ rather than Cat gives span enrichment in parametrized cocomplete objects, use this to deduce universal property for slice category

Proposition I.2.5. *Let* $\mathcal{X}_{/-} : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ *denote the slice functor sending each object (...) and each morphism* $f : X \to Y$ *to its pullback functor* $f^* : \mathcal{X}_{/Y} \to \mathcal{X}_{/X}$*. It is clear that each* f^* *admits a left adjoint* $f_!$ *acting by* $(Z \to X) \mapsto (Z \to X \to Y)$*. Moreover, the* $BC_!$ *condition is satisfied so that* $\mathcal{X}_{/-}$ *is* $\mathcal{X}$*-cocomplete. If* $\mathcal{X}$ *has a terminal object, then* $\mathcal{X}_{/-}$ *is the free* $\mathcal{X}$*-cocomplete* $\mathcal{X}$*-category—in particular, evaluation at* $\mathrm{pt} \to \mathrm{pt} \in \mathcal{X}_{/\mathrm{pt}}$ *induces an equivalence*

$$\mathrm{Fun}^{cc}(\mathcal{X}_{/-}, C) \xrightarrow{\sim} C_{\mathrm{pt}}.$$

Okay, so we've laid down a bit of the categorical underpinnings of what we're interested in. Now let

$$C : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

denote an $\mathcal{X}$-cocomplete $\mathcal{X}$-category. Recall that we are interested in a highly functorial description of homology, which takes as input $M \in C_{\mathrm{pt}}$ and $X \in \mathcal{X}$. One way to recover this is as follows. Because C is an $\mathcal{X}$-cocomplete object in Cat, it lifts to a 2-functor

$$\begin{tikzcd}
\mathrm{Span}_{1.5}(\mathcal{X}) \arrow[r, dashed, "C"] & \mathsf{Cat} \\
\mathcal{X}^{\mathrm{op}} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

This induces a functor

$$\mathcal{X} = \mathrm{End}_{\mathrm{Span}_{1.5}(\mathcal{X})}(*) \to \mathrm{End}_{\mathsf{Cat}}(C_{\mathrm{pt}}).$$

Recall how the lift acts, and hence how this induced functor acts: an object $X \in \mathcal{X}$ is identified with $* \leftarrow X \to *$ and the lift associates to this span its push-pull $(t_X)_!(t_X)^*(-) : C_{\mathrm{pt}} \to C_{\mathrm{pt}}$. This constructs homology $M_C[X]$ as a functor of M, naturally in X.

We may also mate this functor along the exponential adjunction $\mathrm{End}(C_{\mathrm{pt}})^{\mathcal{X}} \cong \mathrm{Fun}(\mathcal{X}, C_{\mathrm{pt}})^{C_{\mathrm{pt}}}$ to produce a functor

$$M_C[-] : C_{\mathrm{pt}} \to \mathrm{Fun}(\mathcal{X}, C_{\mathrm{pt}}).$$

Perhaps by thinking hard enough, you will notice that this is precisely the composition

$$C_{\mathrm{pt}} \xrightarrow{\sim} \mathrm{Fun}^{cc}(\mathcal{X}_{/-}, C) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}(\mathcal{X}, C_{\mathrm{pt}}).$$

We now understand two constructions of $M_C[X]$ which are functorial in M and X. Note that this construction is also functorial in morphisms of $\mathcal{X}$-cocomplete $\mathcal{X}$-categories.

Definition I.2.6. Dually, for an $\mathcal{X}$-complete $\mathcal{X}$-category $C : \mathcal{X}^{\mathrm{op}} \to C$, we may define *$\mathcal{X}$-parametrized cohomology* $M_C^{(-)}$ as the composite

$$C_{\mathrm{pt}} \xrightarrow{\sim} \mathrm{Fun}^{c}((\mathcal{X}_{/-})^{\mathrm{op}}, C) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, C_{\mathrm{pt}}).$$

We have defined our (co)homology functors somewhat more concretely. Now we can look at how integration occurs as a "wrong-way" map on cohomology in the semiadditive situation.

For this, it will be very convenient to exploit another universal property. Recall that the $\mathcal{X}$-category $\mathcal{X}_{/-} : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ admits left adjoints (in the sense that each functor $f^* := \mathcal{X}_{/f}$ admits left adjoints) and these satisfy the $BC_!$-condition, in fact in a free way—this identifies $\mathcal{X}_{/-}$ as the free $\mathcal{X}$-cocomplete $\mathcal{X}$-category [Ben24, §2.2.3]. This was important to our construction of homology, because if we have a terminal object $\mathrm{pt} \in \mathcal{X}$, then for any $\mathcal{X}$-cocomplete category C we could identify

$$\mathrm{Fun}^{cc}_{\mathcal{X}}(\mathcal{X}_{/-}, C) \xrightarrow{\sim} C_{\mathrm{pt}}$$

by sending a functor F to $F_{\mathrm{pt}}(\mathrm{pt})$. Shay's proof relied on the ability to lift to $\mathrm{Span}_{1.5}(\mathcal{X})$ using its universal property as an $\mathcal{X}$-cocomplete object [Ben24, Theorem 2.10] and a relation between the slice category and spans [Ben24, Prop. 2.14].

Remark I.2.7. There's some distinction at play here here between free properties and universal properties. How to explain it lucidly?

We can play the same game with our eye on semiadditivity assuming cocompleteness, as opposed to attacking cocompleteness assuming pointedness. First, let's recall the definition at hand: we say that the $\mathcal{X}$-parametrized $\mathbf{A}$-object $C : \mathcal{X}^{\mathrm{op}} \to \mathbf{A}$ is *$\mathcal{X}$-semiadditive* if it is complete, cocomplete, and for every $f : X \to Y$ the associated [Ben24, Def. 2.15] norm Nm_f is an isomorphism. Just like cocompleteness, there is a universal 2-category with an $\mathcal{X}$-semiadditive object: there exists a functor $\mathcal{X}^{\mathrm{op}} \to U^{\oplus}(\mathcal{X})$ such that precomposition induces an equivalence of 2-categories

$$\mathrm{Fun}(U^{\oplus}(\mathcal{X}), \mathbf{A}) \xrightarrow{\sim} \mathrm{Fun}^{\oplus}(\mathcal{X}^{\mathrm{op}}, \mathbf{A}).$$

In words, the $\mathcal{X}$-semiadditive $\mathbf{A}$-objects can be identified with $\mathbf{A}^{U^{\oplus}(\mathcal{X})}$. One hopes at this point that $U^{\oplus}(\mathcal{X})$ has some spanish construction, similar to how $\mathrm{Span}_{1.5}(\mathcal{X})$ was related to $\mathcal{X}_{/-}$. Surprisingly, life is not so simple. But there is a *comparison* with spans to be made.

Definition I.2.8. Let $\mathcal{X}$ denote an $(\infty, 1)$-category with pullbacks. The 2-category of iterated spans $\mathrm{Span}_2(\mathcal{X})$ has the same objects as $\mathcal{X}$, has spans as 1-morphisms, and has spans between spans as 2-morphisms, e.g. the following diagram depicts a 2-morphism.

$$\begin{tikzcd}
& Z \arrow[dl] \arrow[dr] & \\
X & Q \arrow[u] \arrow[d] & Y \\
& W \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

There is a canonical map $\mathrm{Span}_{1.5}(\mathcal{X}) \to \mathrm{Span}_2(\mathcal{X})$ which acts on 2-morphisms by

$$\begin{tikzcd}
& Z \arrow[dl] \arrow[dr] \arrow[dd] & & & & Z \arrow[dl] \arrow[dr] \arrow[d, no head, "="] & \\
X & & Y \arrow[rr, mapsto] & & X & Z \arrow[d] & Y \\
& W \arrow[ul] \arrow[ur] & & & & W \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Now we have a tower $\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_{1.5}(\mathcal{X}) \to \mathrm{Span}_2(\mathcal{X})$. We would like that the ability to lift through the first and second stage of this tower precisely expresses $\mathcal{X}$-cocompleteness and $\mathcal{X}$-semiadditivity, respectively. It is that latter statement which is not quite correct. But we can say the following.

Proposition I.2.9. *The parametrized object $\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_2(\mathcal{X})$ is $\mathcal{X}$-semiadditive. Hence, there is a classifying factorization through the universal $\mathcal{X}$-semiadditive object*

$$\mathcal{X}^{\mathrm{op}} \to U^{\oplus}(\mathcal{X}) \to \mathrm{Span}_2(\mathcal{X}).$$

Remark I.2.10. Shay explains why the map $U^{\oplus}(\mathcal{X}) \to \mathrm{Span}_2(\mathcal{X})$ should not be an isomorphism in general but conjectures that it is so when $\mathcal{X}$ is truncated [Ben24, Conj. 2.25]. I wonder if this is related to the finiteness requirements for semiadditivity, e.g. why we should not ask for all biproducts but only finite biproducts.

Now let's circle back and recall that we had both the universal 2-category with an $\mathcal{X}$-cocomplete object $\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_{1.5}(\mathcal{X})$ and the free $\mathcal{X}$-cocomplete category $\mathcal{X}_{/-} : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$. In our situation, we have the universal 2-category with an $\mathcal{X}$-semiadditive object $\mathcal{X}^{\mathrm{op}} \to U^{\oplus}(\mathcal{X})$, and we may ask what the free $\mathcal{X}$-semiadditive category is. Curiously, although $\mathrm{Span}_2(\mathcal{X})$ is not the universal 2-category with an $\mathcal{X}$-semiadditive object, the free $\mathcal{X}$-semiadditive category can be realized by a levelwise span construction.

Proposition I.2.11 (Ben-Moshe Prop. 2.28)**.** *If $\mathcal{X}$ is truncated and admits a terminal object* pt*, then* $\mathrm{Span}_1(\mathcal{X}_{/-}) : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ *is the free $\mathcal{X}$-semiadditive $\mathcal{X}$-category. In particular, if C is an $\mathcal{X}$-semiadditive $\mathcal{X}$-category, then evaluation at* $\mathrm{pt} \to \mathrm{pt}$ *induces an equivalence*

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}_1(\mathcal{X}_{/-}), C) \xrightarrow{\sim} C_{\mathrm{pt}}.$$

We are finally ready to do to integration what we did to (co)homology. Namely, just as we found a convenient parametrized language to describe $\mathcal{X}$-(co)completeness and decategorified it into highly functorial (co)homology functors, we may now take our language for $\mathcal{X}$-semiadditivity and decategorify it into highly functorial integration functors.

The first thing to realize is that in the presence of semiadditivity, cohomology possesses a spanish bivariance.

Definition I.2.12. Suppose that $\mathcal{X}$ is truncated and that $C : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ is $\mathcal{X}$-semiadditive. We define the *$\mathcal{X}$-cohomology with integration* functor as the composite

$$C_{\mathrm{pt}} \xrightarrow{\sim} \mathrm{Fun}^{\oplus}(\mathrm{Span}_1(\mathcal{X}_{/-}), C) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}(\mathrm{Span}_1(\mathcal{X}), C_{\mathrm{pt}}).$$

We denote by $M_C^{(-)}$ the image of M.

Definition I.2.13. For $f : X \to Y$ in $\mathcal{X}$, we define the *integral along f* as the image of the span $X = X \xrightarrow{f} Y$

$$\int_f : M_C^X \to M_C^Y.$$

Remark I.2.14. Restricting integrable cohomology $M_C^{(-)}$ along $\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_1(\mathcal{X})$ reproduces cohomology. Likewise, restricting along $\mathcal{X} \to \mathrm{Span}_1(\mathcal{X})$ reproduces homology. In particular, integration along f is precisely the morphism induced on homology. I still haven't precisely unwound this from [Ben24, Prop. 3.4, Def. 3.7].

I think that's enough for this entry. I'll end with two comments. The first is that integration can be further decategorified by restricting to $\mathrm{End}_{\mathrm{Span}_1(\mathcal{X})}(\mathrm{pt}) \cong \mathcal{X}$ to define cardinality. The second is that everything in sight is in fact *functorial in the target C* and also interacts nicely under *change of parameters in $\mathcal{X}$*, and this is important to studying the local systems business, in particular the higher character maps.

I.3 (1/29) Unwinding definitions

I have it in mind to study the integration map on E_n-cohomology in a highly functorial setting. The purpose of my last two entries was to orient myself for this. Specifically, I want to consider

$$\mathcal{X} = \mathsf{S}_{\pi\text{-fin}}, \quad C_{\mathrm{pt}} = \mathsf{Mod}_{E_n}(\mathrm{Sp}_{K(n)}), \quad M = E_n \in C_{\mathrm{pt}}, \quad \text{and} \quad C = \mathsf{Mod}_{E_n}(\mathrm{Sp}_{K(n)})^{(-)}.$$

Note that C is the $\mathcal{X}$-category induced by C_{pt}, and that C is $\mathcal{X}$-semiadditive because C_{pt} is ∞-semiadditive. Then we may consider the integrable cohomology $M_C^{(-)} = E_n^{(-)}$ for which a map $f : X \to Y$ induces

$$\int_f : E_n^X \to E_n^Y.$$

This is a morphism in $C_{\mathrm{pt}} = \mathsf{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$. I made a small mistake while defining it in my last entry, where I originally thought I was correcting a small typo of Shay's.

Let's take a few steps back. Recall: because our C is $\mathcal{X}$-semiadditive, we can naturally associate to $M \in C_{\mathrm{pt}}$ an *integrable cohomology* functor

$$M_C^{(-)} : \mathrm{Span}_1(\mathcal{X}) \to C_{\mathrm{pt}}.$$

Proposition I.3.1. *Restricting $M_C^{(-)}$ along the inclusion $\mathcal{X}^{\mathrm{op}} \to \mathrm{Span}_1(\mathcal{X})$ sending a morphism $f^{\mathrm{op}} : Y \to X$ to $Y \xleftarrow{f} X = X$ recovers cohomology $M_C^{(-)} : \mathcal{X}^{\mathrm{op}} \to C_{\mathrm{pt}}$.*

Proof. This is a matter of unwinding definitions. As that is kind of the point of this entry, let's unwind. Recall that $M_C^{(-)}$ is the image of M under

$$C_{\mathrm{pt}} \xrightarrow{\sim} \mathrm{Fun}^{\oplus}(\mathrm{Span}_1(\mathcal{X}_{/-}), C) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}(\mathcal{X}, C_{\mathrm{pt}}).$$

The $\mathcal{X}$-category $\mathrm{Span}_1(\mathcal{X}_{/-}) : \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$ sends an object X to $\mathrm{Span}_1(\mathcal{X}_{/X})$, and a morphism $f : X \to Y$ to the pullback $f^* : \mathrm{Span}_1(\mathcal{X}_{/Y}) \to \mathrm{Span}_1(\mathcal{X}_{/X})$. That stated, let $\tilde{M} : \mathrm{Span}_1(\mathcal{X}_{/-}) \to C$ denote the image of M under the first identification. This is not just an assemblage of fiberwise functors—it is an $\mathcal{X}$-functor, so it respects reindexing, in particular for $t_X : X \to \mathrm{pt}$ we have

$$\tilde{M}_X(t_X)^* \cong (t_X)^*\tilde{M}_{\mathrm{pt}} \quad \text{as functors} \quad \mathrm{Span}_1(\mathcal{X}_{/\mathrm{pt}}) \to C_X.$$

It is clear that $(t_X)^*(\mathrm{pt} \to \mathrm{pt}) = \mathrm{id}_X$. On the other hand, the universal property of $\mathrm{Span}_1(\mathcal{X}_{/-})$ is such that $\tilde{M}_{\mathrm{pt}}(\mathrm{pt} \to \mathrm{pt}) = M$. Reindexing then forces $\tilde{M}_X(\mathrm{id}_X) \cong (t_X)^*(M)$, thus

$$(t_X)_*(t_X)^*(M) \cong (t_X)_*\tilde{M}_X(\mathrm{id}_X) = \tilde{M}_{\mathrm{pt}}(X \to \mathrm{pt}).$$

This verifies $\tilde{M}_{\mathrm{pt}}(X \to *) = M_C^X$. Because the integrable cohomology $M_C^{(-)}$ acts on X by evaluating $\tilde{M}_{\mathrm{pt}}$ on $X \to *$, this proves our claim on objects.

We must next check the effect on morphisms. Let's first explain the precise definition of the morphism we seek to identify as that induced by integrable cohomology: given $f : X \to Y$, we have $(t_Y) \circ f = t_X$, hence $(t_X)^* \cong f^*(t_Y)^*$ and $(t_X)_* \cong (t_Y)_* f_*$. (You see something like this in e.g. [Ben24, Prop. 3.4].) Then we get a canonical identification $M_C^X = (t_X)_*(t_X)^*(M) \cong (t_Y)_* f_* f^*(t_Y)^*(M)$. The map on cohomology occurs as the composition

$$(t_Y)_*(t_Y)^*(M) \xrightarrow{(t_Y)_*(u_{*,t_Y^* M})} (t_Y)_* f_* f^*(t_Y)^*(M) \cong (t_X)_*(t_X)^*(M).$$

This is the morphism in C_{pt} which we would like to identify as the image of $Y \xleftarrow{f} X \to X$ under $\tilde{M}_{\mathrm{pt}}$. Well, this is what we want more-or-less by definition: a general span is acted upon by restricting along the left leg (implemented via the unit) then integrating along the right leg (implemented via the "wrong-way unit" which only generally exists semiadditively). In this case, since the right leg is just id_X, this is just the same map we wrote out above. ☐

Likewise, we can recover homology by restricting along $\mathcal{X} \to \mathrm{Span}_1(\mathcal{X})$. Also likewise, it will be useful to recall a bunch of definitions as we explain this claim.

Proposition I.3.2. *Restricting $M_C^{(-)}$ along the inclusion $\mathcal{X} \to \mathrm{Span}_1(\mathcal{X})$ sending a morphism $f : X \to Y$ to $X = X \xrightarrow{f} Y$ recovers homology $M_C[X] : \mathcal{X} \to C_{\mathrm{pt}}$.*

Proof. First, let's recall the *ordinary* form of homology we are seeking. We defined ordinary homology $M_C[-]$ as the image of M under

$$C_{\mathrm{pt}} \xrightarrow{\sim} \mathrm{Fun}^{cc}(\mathcal{X}^{/-}, C) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}(\mathcal{X}, C_{\mathrm{pt}}).$$

Similar to how we checked the value of $M_C^{(-)}$ on objects above, you can check that the image of M under the first identification $\widetilde{M}_{\mathrm{pt}} : \mathcal{X}^{/*} \to C_{\mathrm{pt}}$ sends $X \to *$ to $(t_X)_! t_X^*(M)$, whence $M_C[X] = (t_X)_!(t_X)^*(M)$ as desired. Recall that we also checked this earlier by checking the corresponding claim for the exponential mate of $C_{\mathrm{pt}} \to \mathrm{Fun}(\mathcal{X}, C_{\mathrm{pt}})$: we lift $C : \mathrm{Span}_{1.5}(\mathcal{X}) \to \mathsf{Cat}$ using that C is $\mathcal{X}$-cocomplete and the free property of $\mathrm{Span}_{1.5}(\mathcal{X})$, where $M_C[X]$ is identified with the image of $X \in \mathcal{X} \cong \mathrm{End}_{\mathrm{Span}_{1.5}(\mathcal{X})}(\mathrm{pt})$ under the induced functor $\mathrm{End}_{\mathrm{Span}_{1.5}(\mathcal{X})}(\mathrm{pt}) \to \mathrm{End}(C_{\mathrm{pt}})$, and here we know the lift acts by push-pulling along $\mathrm{pt} \leftarrow X \to \mathrm{pt}$, verifying the claim on objects.

Let's also verify what ordinary homology $M_C[-]$ does to the morphism $f : X \to Y$. As before, we have two options: we can check $M_C[-]$ or we can check its exponential mate $(-)_C[X \to Y]$. I'm not really sure how to do it directly, and Shay already explains how to check the mate [Ben24, Prop. 3.4], so let me just explain that. We are asking, having taken the exponential mate, about the image of f under $\mathcal{X} \cong \mathrm{End}_{\mathrm{Span}_{1.5}(\mathcal{X})}(\mathrm{pt}) \to \mathrm{End}(C_{\mathrm{pt}})$, specifically what it does to the object M. Observe that f is the following composite of three 2-morphisms:

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[d, "="] \arrow[dr, "="] & & X \arrow[dl, "f"'] \arrow[d, "f"] \arrow[dr, "f"] & & Y \arrow[dl, "="'] \arrow[d, "="] \arrow[dr] & \\
\mathrm{pt} & {} & Y & {} & Y & {} & \mathrm{pt} \\
 & Y \arrow[ul] \arrow[ur, "="'] & & Y \arrow[ul, "="] \arrow[ur, "="'] & & Y \arrow[ul, "="] \arrow[ur] &
\end{tikzcd}$$

in the sense that after forming all the pullbacks, the total square is just

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[d, "f"] \arrow[dr] & \\
\mathrm{pt} & {} & \mathrm{pt} \\
 & Y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

See that the "upper half" of the big diamond of pullbacks expresses a 2-morphism $(\mathrm{pt} \leftarrow X \to \mathrm{pt}) \to (\mathrm{pt} \leftarrow Y \to Y \leftarrow X \to Y \leftarrow Y \to \mathrm{pt})$ in $\mathrm{Span}_{1.5}(\mathcal{X})$. This is a 1-morphism in $\mathrm{End}_{\mathrm{Span}_{1.5}(\mathcal{X})}(\mathrm{pt})$ whose image under the mate of $M_C[-]$ is the canonical identification

$$(t_X)_!(t_X)^* \xrightarrow{\sim} (t_Y)_! f_! f^* (t_Y)^* \quad \text{in} \quad \mathrm{End}(C_{\mathrm{pt}}).$$

The composition of three 2-morphisms now inserts the counit $c_!^f : f_! f^* \to \mathrm{id}_{C_Y}$ to obtain

$$(t_X)_!(t_X)^* \xrightarrow{\sim} (t_Y)_! f_! f^* (t_Y)^* \xrightarrow{(t_Y)_! c_!^f{}_{(t_Y)^*}} (t_Y)_!(t_Y)^*.$$

Summarizing: we were interested in the image $M_C[-]$ of M under $C_{\mathrm{pt}} \to \mathrm{Fun}(\mathcal{X}, C_{\mathrm{pt}})$, so we exhibited the exponential mate $\mathcal{X} \to \mathrm{End}(C_{\mathrm{pt}})$ at the morphism $f : X \to Y$ as the above morphism in $\mathrm{End}(C_{\mathrm{pt}})$, and we can now evaluate at M to find that $M_C[f : X \to Y]$ is the morphism

$$M_C[X] = (t_X)_!(t_X)^*(M) \xrightarrow{\sim} (t_Y)_! f_! f^* (t_Y)^*(M) \xrightarrow{(t_Y)_! c_!^f{}_{(t_Y)^*}} (t_Y)_!(t_Y)^*(M) = M_C[Y].$$

What's left is to verify the claim that this is what you get by evaluating the lift $M_C^{(-)} : \mathrm{Span}_{1.5}(\mathcal{X}) \to C_{\mathrm{pt}}$ at the image $X = X \xrightarrow{f} Y$ of f under $\mathcal{X} \to \mathrm{Span}_{1.5}(\mathcal{X})$. (...) $\square$

Now, because C is $\mathcal{X}$-semiadditive, there is a canonical isomorphism $f_! \xrightarrow{\sim} f_*$ and thus $M_C[X] \cong M_C^X$. The image of $X = X \xrightarrow{f} Y$ under integrable cohomology $M_C^{(-)}$ is therefore a morphism

$$M_C^X = (t_X)_*(t_X)^*(M) \cong (t_X)_!(t_X)^*(M) \xrightarrow{\sim} (t_Y)_! f_! f^*(t_Y)^*(M) \xrightarrow{(t_Y)_! c_!^f (t_Y)^*} (t_Y)_!(t_Y)^*(M) \cong (t_Y)_*(t_Y)^*(M) = M_C^Y.$$

And there is our parametrized integration morphism $\int_f : M_C^X \to M_C^Y$.

Bibliography

[CSY22] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. "Ambidexterity in chromatic homotopy theory". In: *Inventiones mathematicae* 228.3 (2022), pp. 1145–1254. DOI: 10.1007/s00222-022-01099-9.

[Mac22] Andrew W. Macpherson. "A bivariant Yoneda lemma and $(\infty, 2)$-categories of correspondences". In: *Algebraic & Geometric Topology* 22 (2022), pp. 2689–2774. DOI: 10.2140/agt.2022.22.2689.

[Ben24] Shay Ben-Moshe. *Higher Semiadditivity in Transchromatic Homotopy Theory*. 2024. DOI: 10.48550/arXiv.2411.00968. arXiv: 2411.00968 [math.AT]. URL: https://arxiv.org/abs/2411.00968.