514 COMPLEX ALGEBRAIC GEOMETRY

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ABSTRACT. Notes for MATH 512, Complex Algebraic Geometry, taught by Sheldon Katz, Fall 2022. I stopped trying to simultaneously TEX lecture and reading notes after Week 2. I found that using the readings to clarify the lecture notes was a better approach. I also am not including Sheldon's assigned problems.

CONTENTS

0.1. Voisin.

0.1.1. *One complex variable.* Let $U \subseteq \mathbb{C}$ be open. Then we consider the differential forms

$$dz = dx + i\,dy, \quad d\bar{z} = dx - i\,dy.$$

Both are elements of $\mathrm{Hom}_{\mathbb{R}}(T_U, \mathbb{C}) \cong \Omega_{U,\mathbb{R}} \otimes \mathbb{C}$. These form a basis for $\Omega_{U,\mathbb{R}} \otimes \mathbb{C}$. (Look at $dz - d\underline{z}$ and $dz + d\underline{z}$.)

Let $f : U \to \mathbb{C}$ be C^1. The differential form df_u is a map $T_{U,u} \cong \mathbb{C} \to \mathbb{C}$. We may uniquely write

$$df_u = f_z(u)dz + f_{\bar{z}}(u)d\bar{z},$$

For some continuous $f_z, f_{\bar{z}}$. One has (or may take as definitions) that $f_z(u) = \frac{\partial f}{\partial z}(u)$ and $f_{\bar{z}}(u) = \frac{\partial f}{\partial \bar{z}}(u)$. Note that $\frac{\partial f}{\partial \bar{z}} = 0$ is the Cauchy-Riemann equation, and we say that f is **holomorphic** if f satisfies $\frac{\partial f}{\partial \bar{z}}(u) = 0$ for all $u \in U$. What does holomorphicity say about df_u?

Lemma 0.1. *If $f : U \to \mathbb{C}$ is C^1, then f satisfies $\frac{\partial f}{\partial \bar{z}}(u) = 0 \iff df_u$ is $\mathbb{C}$-linear.*

To demonstrate that this is a useful perspective, we shall use it to prove the following.

Lemma 0.2. *If f and g are holomorphic, then so are $f + g$ and fg. Furthermore, f is nonvanishing, then $1/f$ is holomorphic. And if f has image in the domain of g, then $g \circ f$ is holomorphic.*

Proof. Since d is linear, if f, g are holomorphic then $d_{f+g} = d(f + g)_u = df_u + dg_u$, which is the sum of two $\mathbb{C}$-linear functions and hence $\mathbb{C}$-linear. So $f + g$ is holomorphic. One shows that fg is linear likewise.

For $g \circ f$, the chain rule says that $d(g \circ f)_u = dg_{f(u)} \circ df_u$. This is a composition of two $\mathbb{C}$-linear functions, hence it is $\mathbb{C}$-linear. So $g \circ f$ is holomorphic.

Finally, we can show that if f is nonvanishing then $1/f$ is holomorphic; this is just a corollary of the fact that compositions are holomorphic, along with the fact that $z \mapsto 1/z$ is holomorphic on $\mathbb{C}^\times$.

0.1.2. *Stokes' theorem; Cauchy's formula as an application.* Let M be a k-manifold and $\phi : M \to U$ a C^1 map. If α is a continuous k-form on U, then $\phi^*\alpha$ is such a form on M. Moreover, if M is compact and oriented then that pullback can be integrated.

Likewise, if α is a C^1 $(k-1)$-form on U, then we can integrate $\phi^*\alpha$ on ∂M, and we can integrate $\phi^* d\alpha = d\phi^*\alpha$ on M.

(Stoke's Theorem) One has

$$\int_M \phi^* d\alpha = \int_{\partial M} \phi^*\alpha.$$

In particular, if α is closed, i.e. $d\alpha = 0$, then $\int_{\partial M} \phi^*\alpha = 0$.

To get Cauchy's formula out of Stokes' theorem, we need the following fact: *if $f : U \to \mathbb{C}$ is holomorphic, then $f\,dz$ is a closed form.* In particular, $f/(z - z_0)\,dz$ is closed on $U - \{z_0\}$.

Now we can do the following. Let $D \subset U$ be a closed disk, and for $z_0 \in U$ let D_ϵ denote the open disk about z_0 of radius ϵ, small enough to fit in D. Stokes' theorem applied to $D - D_\epsilon$ gives us

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i}\int_{|z - z_0| = \epsilon} \frac{f(z)}{z - z_0} dz. \tag{1}$$

A short analytic argument shows that the RHS of Equation (1) converges to $f(z_0)$ as $\epsilon \to 0$. Taking $\epsilon \to 0$, we get Cauchy's formula.

0.1.3. *Several complex variables.* For open $U \subset \mathbb{C}^n$, let $f : U \to \mathbb{C}$ be C^1. We can as before identify $T_{U,u} \cong \mathbb{C}^n$, and define holomorphic functions of several variables accordingly.

Definition 0.1. The function f is said to be **holomorphic** if for every $u \in U$, the differential $df_u : T_u \cong \mathbb{C}^n \to \mathbb{C}$ is $\mathbb{C}$-linear.

And we have a lemma similar to the one before. This lemma, together with Stokes' theorem and Cauchy's formula for several variables, is used to prove the following theorem.

Lemma 0.3. *If f is holomorphic, then $f\, dz_1 \wedge \cdots \wedge dz_n$ is a closed form.*

Theorem 0.1. *Let $f : U \to \mathbb{C}^n$ be C^1. TFAE:*

(1) f is holomorphic.
(2) Each point $z_0 \in U$ has a neighborhood in which f admits a power series expansion of the form

$$f(z_0 + z) = \sum_I a_I z^I,$$

Where I ranges over n-tuples of positive integers and $z^I := z_1^{i_1} \dots z_n^{i_n}$, and such that the coefficients α_I satisfies the property: there exist $R_i > 0$ such that

$$\sum_I |\alpha_I| r^I$$

Converges for every $r_1 < R_1, \dots, r_n < R_n$.
(3) If $D = \{(z'_1, \dots, z'_n) : |z'_i - a_i| \le \alpha_i\}$ is a polydisk within U, then for every $z \in D^0$ we have

$$f(z) = (2\pi i)^{-n} \int_{|z'_i - a_i| = \alpha_i} f(z') \frac{dz'_1}{z'_1 - z_1} \wedge \cdots \wedge \frac{dz'_n}{z'_n - z_n},$$

Where the integral is taken over the product of circles.

One uses this to prove, for instance, the maximum principle and identity theorem for scv. Also, more subtle facts like the Riemann extension theorem and Hartog's theorem; these were already covered in class, see Section 1.2.

0.1.4. *The $\bar\partial$-Poincaré Lemma.* We briefly introduced the operator $\bar\partial$ on Wednesday, when we stated Lemma 1.1. We defined

$$\bar\partial f := \frac{\partial f}{\partial \bar z} d\bar z.$$

Theorem 0.2 ($\bar\partial$-Poincaré Lemma)**.** *Let f be a C^k function defined on $U \subset \mathbb{C}$. Then, locally, there exists a C^k function g such that*

$$\frac{\partial g}{\partial \bar z} = f.$$

Proof. WLOG, we may assume that f has compact support, as this is a local result; therefore it extends to a function that is defined and C^k on $\mathbb{C}$. I claim that

$$g(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\mathbb{C} - D(\epsilon, z)} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta$$

Is what we want. Changing variables with $\zeta' = \zeta - z$, we define

$$g_\epsilon(z) := \frac{1}{2\pi i} \int_{\mathbb{C} - D(\epsilon, 0)} \frac{f(\zeta' + z)}{\zeta'} d\zeta' \wedge d\bar\zeta'.$$

As $\epsilon \to 0$, we have that $g_\epsilon(z)$ converges uniformly to $g(z)$. Differentiating w.r.t. $d\bar{z}$, we get

$$\frac{\partial g_\epsilon}{\partial \bar{z}} = \frac{1}{2\pi i} \int_{\mathbb{C}-D(\epsilon,0)} \frac{\partial f(\zeta' + z)}{\partial \bar{z}} \frac{d\zeta' \wedge d\bar{\zeta}'}{\zeta'}.$$

Since $\frac{\partial f(\zeta'+z)}{\partial \bar{z}}$ is C^{k-1}, we have that $\frac{\partial g_\epsilon}{\partial \bar{z}}$ converges uniformly. We conclude that g is at least C^1; inducting (see Voisin) we get that g is actually C^k.

It remains to show the desired equality. First note that away from z, i.e. on $\mathbb{C} - D_\epsilon$,

$$d(f\frac{d\zeta}{\zeta - z}) = -\frac{\partial f}{\partial \bar{\zeta}}(\zeta)\frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

(See Voisin p. 36; not complicated.) Now Stokes' says that

$$\frac{1}{2\pi i} \int_{C-D(\epsilon,z)} \frac{\partial f}{\partial \bar{\zeta}}(\zeta)\frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} = \frac{1}{2\pi i} \int_{\partial D(\epsilon,z)} f(\zeta)\frac{d\zeta}{\zeta - z}.$$

As $\epsilon \to 0$, we know the RHS converges to $f(z)$. And the LHS converges to $\frac{\partial g}{\partial \bar{z}}(z)$. $\square$

1. Week 1 Lecture Notes

1.1. **(8/24) Wednesday.** I would call today 'foreshadowing day.'

Lemma 1.1. *Recall that* $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$. *We write*

$$\partial f := \frac{\partial f}{\partial z}dz, \quad \bar{\partial} f := \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Then $\partial^2 = 0, \bar{\partial}^2 = 0$, *and* $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

The operator $\bar{\partial}$ will be important later; today, we just proved $\frac{\partial g}{\partial \bar{z}} = f$ is always locally solvable, see Section 0.1.4.

Now let's talk several variables. The tangent space $T_p\mathbb{C}^n$ has an $\mathbb{R}$-basis given by the $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$. After complexifying, $T_p\mathbb{C}^n$ has a $\mathbb{C}$-basis given by the $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{z_i}}$. Now we define the **holomorphic tangent space** (see GH p. 16)

$$T_z^{1,0}\mathbb{C}^n := \text{span}_{\mathbb{C}}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\} \subset T_z\mathbb{C}^n \otimes \mathbb{C}.$$

"Take the real part" defines an isomorphism $T_z^{1,0}\mathbb{C}^n \to T_z\mathbb{C}^n$. Now, observe that multiplication by i defines an automorphism of $T_z^{1,0}\mathbb{C}^n$; let J denote the induced automorphism of $T_z\mathbb{C}^n$. Then we have that

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}, \quad J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}, \quad J^2 = -1.$$

This will be important later. Now one more unrelated fact...

Definition 1.1. f is **holomorphic** if $\frac{\partial f}{\partial \bar{z}_i} = 0$ for all i.

Definition 1.2. f is **analytic** if for all $z_0 \in U \subseteq \mathbb{C}^n$, $\exists$nhbd V of z_0 in U on which $f(z) = \sum a_I z^I$, i.e., f is equal to some convergent power series.

Lemma 1.2. *holomorphic* $\iff$ *analytic.*

1.2. **(8/26) Friday.** We have a few theorems regarding a single complex variable which generalize. They are proven as applications of Theorem 0.1. In what follows, let $U \subset \mathbb{C}^n$ be open, and let f, g be holomorphic on U.

Lemma 1.3. *(Max Principle) If $|f|$ obtains its maximum at some point $u \in U$, then f is locally constant near u.*

Lemma 1.4. *(Identity Thm) If U is connected and $f = g$ on a nonempty open $V \subset U$, then $f = g$ on U.*

Sheldon: "Now we can have some fun." What he meant is that we can now prove **another** several variable version of a single variable result. The fun part is that this fact will have related results which demonstrate rich theory in several complex variables which is not visible in a single variable.

Lemma 1.5. *Suppose f is holomorphic on U except possibly at a single point z_1; without loss of generality, suppose U contains 0 and that $z_1 = 0$. In other words, suppose $f \in \mathcal{O}(U - \{0\})$. If f is bounded near 0, then f extends to $\mathcal{O}(U)$.*

We proved the above lemma. It was good. Unfortunately Sheldon disfigured some notation, so I won't reproduce it here. I think it is in Griffiths and Harris.

Now here is a similar result, and a generalization:

Lemma 1.6. *If $g \in \mathcal{O}(U)$ and $f \in \mathcal{O}(U - \{z : g(z) = 0\})$, and f is locally bounded near $\{z : g = 0\}$, then f extends to $\mathcal{O}(U)$.*

(Hartog's Theorem) Let $\Delta_r \subset \mathbb{C}^n$ be a polydisk. Let $\Delta_{r'}$ be a polydisk within Δ_r. If $f \in \mathcal{O}(\Delta_r - \Delta_{r'})$, then f extends to $\mathcal{O}(\Delta_r)$.
An important corollary: **If $n > 1$ and $f \in \mathcal{O}(\Delta^\times \subset \mathbb{C}^n)$, then f extends to $\mathcal{O}(\Delta)$.**

1.3. Voisin.

1.3.1. *Smooth manifolds and vector bundles.* I'll just assume as known C^k and **smooth manifolds** as well as C^k **functions on C^k or smooth manifolds**. Defined in lecture, see Section 2.2. Now, a definition not found there:

Definition 1.3 (Vector bundle)**.** Let X be a smooth m-manifold. Let K denote either $\mathbb{R}$ or $\mathbb{C}$. Consider the following data (all maps being continuous):

(1) A space E and a surjection $\pi : E \to X$ (we write $E_x := \pi^1(x)$),
(2) A K-vector space structure on $\pi^{-1}(x)$ for every $x \in X$,
(3) A cover $\{U_i\}$ of X with **local trivializations** $\tau_i : \pi^{-1}(U_i) \to U_i \times K^n$ such that
   - The trivializations τ_i are homeomorphisms,
   - The triangle commutes, i.e., $\mathrm{proj}_{U_i} \circ \tau_i = \pi$,
   - The transitions $\tau_i \circ \tau_j^{-1}$ are K-linear when restricted to each fiber $u \times K^n$.

All this data defines a **(real or complex) topological vector bundle** over X. For each pair i, j, note that $\tau_i \circ \tau_j^{-1}$ is an automorphism of $(U_i \cap U_j) \times k^n$, and we asked that it be k-linear on each $u \times k^n$. Thus, we may write

$$\begin{aligned} \tau_i \circ \tau_j^{-1} : (U_i \cap U_j) \times k^n &\to (U_i \cap U_j) \times k^n \\ (u, v) &\mapsto (u, g_{ij}(u)v), \end{aligned}$$

For some continuous $g_{ij} : U_i \cap U_j \to GL_m(K)$. Call g_{ij} the **transition matrices**. Written as a matrix of functions of u, these functions are continuous. If $K = \mathbb{R}$ and the transition matrices are C^k, we say that **the vector bundle E is equipped with a C^k differentiable structure**. Any real person would just take $k = \infty$ and call it a **smooth vector bundle**.

Remark 1.1. I think that (2) is vacuous. That is, I think the transition functions canonically induce a K-vector space structure on fibers.

Definition 1.4 (Section)**.** Let $\pi : E \to X$ be a vector bundle. A **global section** is a map $\sigma : X \to E$ such that $\pi \circ \sigma = id$.

Definition 1.5 (Bundle map)**.** Let π_E, π_F be two bundles over X. A **morphism of vector bundles** or **bundle map** between them is a continuous map $\phi : E \to F$ such that $\pi_E = \phi \circ \pi_F$ and such that ϕ is linear on every fiber.

Definition 1.6 (Trivial bundle)**.** A vector bundle π is **trivial** if it has a global trivialization $E \cong X \times k^n$. Equivalently: (1) It admits a bundle isomorphism to the trivial bundle, (2) It admits n global sections which span the fiber E_x at every x.

Definition 1.7 (Dual bundle)**.** Let $\pi : E \to X$ denote a vector bundle. Denote by E^* its **dual bundle**, defined by $(E^*)_x := (E_x)^*$ with trivializations coming from E's. A consequence: the transition matrices of E^* are the inverse of the transposes of E's.

Definition 1.8 (Exterior powers)**.** Let $\pi : E \to X$ denote a vector bundle. Denote by $\wedge^k E$ its k-th **exerior power**, defined by $(\wedge^k E)_x :=$ alternating k-linear forms on E_x^*.

There is a systematic way to turn constructions of vector spaces (e.g., direct sum, tensor product, dualization, taking exterior powers) into constructions of bundles; Sheldon mentioned this briefly, and Voisin does not mention it.

1.3.2. *The tangent bundle.* Let X denote a smooth manifold. The **real tangent bundle** $T\mathbb{R}_X$ is a smooth bundle over X. Write $C^\infty(X)_x$ for the ring of (germs of) C^∞ functions $U \to \mathbb{R}$ defined near x; that is,

$$C^\infty(X)_x := \coprod_{U \subseteq X \,:\, x \in U} C^\infty(U)/(f \sim g \text{ if } f = g \text{ on some } V \subseteq U \text{ containing } x).$$

And we call a map $F : C^\infty(X)_x \to \mathbb{R}$ a **derivation** if it is $\mathbb{R}$-linear and satisfies $F(fg) = F(f)g(x) + f(x)F(g)$. Now we can define the tangent bundle.

Definition 1.9. (Real tangent bundle) Let X be a smooth n-manifold. Its **(real) tangent bundle** $T\mathbb{R}_X$ has fiber

$$T\mathbb{R}_{X,x} := \{\text{derivations } C^\infty(X)_x \to \mathbb{R}\}.$$

The total space (as a set) is the disjoint union of the $T\mathbb{R}_{X,x}$. The projection π is the obvious one. Its trivializations are defined as follows. Take a cover $\{U_i, \phi_i\}$ of X, then define $\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ by

$$(x \in U_i, \sum t_i \frac{\partial}{\partial x_i}\big|_x) \mapsto (x, t_1, \ldots, t_n).$$

This is a smooth $2n$-dimensional manifold and a rank n bundle.

One can describe points of $T\mathbb{R}_X$ using **jets** or **derivations**. Sheldon seemed to not care about jets. Our definition of $T\mathbb{R}_X$ already tells us how to think in terms of derivations.

Definition 1.10 (Real differential form)**.** Consider the bundle $\wedge^k T\mathbb{R}_X^*$. At x, its fiber is $\wedge^k(T\mathbb{R}_{X,x}^*)$, the space of alternating k-linear maps on $T\mathbb{R}_{X,x}$. A **(real) differential k-form** on X is a section of $\wedge^k T\mathbb{R}_X^*$.

1.3.3. *Complex manifolds.* Let X be a smooth $2n$-manifold. A **complex structure** on X is a cover $\{U_i\}$ with diffeomorphisms $\phi_i : U_i \to V_i \subseteq \mathbb{C}^n$ whose transitions $\phi_i \circ \phi_j^{-1}$ are holomorphic. If X has a complex structure, we say it has **complex dimension** n, and $f : X \to \mathbb{C}$ is called holomorphic if $f \circ \phi_i^{-1}$ is holomorphic for all i.

Definition 1.11 (Holomorphic bundle)**.** Let X be a complex manifold, $\pi : E \to X$ a smooth, complex vector bundle over it. We say π has **holomorphic structure** if we have trivializations $\tau_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}^n$ such that the transition matrices for $\tau_i \circ \tau_j^{-1}$ have holomorphic coefficients.

Definition 1.12 (Holomorphic tangent bundle)**.** Let X be a complex manifold. It has a **holomorphic tangent bundle** T_X constructed in a way similar to how we defined $T\mathbb{R}_X$; the result is a holomorphic bundle over X. Here's the construction.

Let $\mathcal{O}(U)$ denote the ring of holomorphic functions $U \to \mathbb{C}$. Let $\mathcal{O}_{X,x}$ denote the ring of germs of holomorphic functions near x; that is,

$$\mathcal{O}_{X,x} := \coprod_{U \subseteq X \,:\, x \in U} \mathcal{O}(U) \Big/ (f \sim g \text{ if } f = g \text{ on some open } V \text{ containing } x).$$

Then we define $T_{X,x}$ to be the space of **complex derivations** on $\mathcal{O}_{X,x}$, i.e., linear[1] maps $\mathcal{O}_{X,x} \to \mathbb{C}$ satisfying the Leibniz identity.

[1] Real or complex linear?

1.3.4. *Integrability of almost complex structures.* Let X be a complex manifold. Then for covering U_i, we have holomorphic charts $\phi_i : U_i \to \mathbb{C}^n$. Somehow, via pushing forward along ϕ_i, we may identify $T\mathbb{R}_{U_i} \cong U_i \times \mathbb{C}^n$. Then $1 \times i$ induces an automorphism of $T\mathbb{R}_{U_i}$. Moreover, since the transitions ϕ_{ij} are holomorphic, the induced complex structure on tangent spaces is independent of i, therefore these J_i agree on overlap. The result is an map $J : T\mathbb{R}_X \to T\mathbb{R}_X$ such that $J^2 = -1$.

Definition 1.13 (Almost complex structure)**.** An **almost complex structure** on a smooth manifold X is an endomorphism $I : T\mathbb{R}_X \to T\mathbb{R}_X$ such that $I^2 = -1$. Such is equivalent to the structure of a complex vector bundle on $T\mathbb{R}_X$.

Definition 1.14. An almost complex structure I is called **integrable** if it is induced by a complex structure on X.

Now we ask how, in the presence of a.c. structures, we can relate $T\mathbb{R}_X$ and T_X. (Really, the holomorphic tangent bundle T_X was not defined for a.c. manifolds—so there is something interesting going on!)

Let X be a smooth manifold. Denote by $T\mathbb{C}_X$ the complexified real tangent space $T\mathbb{R}_X \otimes \mathbb{C}$. If X is in fact complex, then $T_X \subset T\mathbb{C}_X$, for a chart U_i has its holomorphic tangent space T_{U_i} generated by $\frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i})$ which sits inside $T\mathbb{C}_X$.

Now let (X, J) be an a.c. manifold. Then $T\mathbb{C}_X$ has a complex subbundle $T_X^{1,0}$ of eigenvectors of J for the eigenvalue $+i$. The 'real part' defines an *isomorphism*[2] $T_X^{1,0} \to T\mathbb{R}_X$, and furthermore identifies the operators i on $T_X^{1,0}$ and J on $T\mathbb{R}_X$.

Now let X be complex, U_i a chart with $z_j = x_j + iy_j$ local complex coordinates. Then $T\mathbb{R}_{U_i}$ has basis $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}_j$. The induced complex structure J acts by $\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial y_j}$, $\frac{\partial}{\partial y_j} \mapsto -\frac{\partial}{\partial x_j}$. Thus, a $+i$ eigenvector of J_{U_i} (at a point) is a multiple of $\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}$. Patching together reveals that $T_X^{1,0} = T_X$.

Proposition 1.1. *Summarizing: A complex manifold X has a (canonical) almost complex structure, and the resulting complex vector subbundle $T_X^{1,0} \subset T\mathbb{C}_X$ is equal to T_X.*

Definition 1.15 (Antiholomorphic tangent space)**.** Let (X, J) be an a.c. manifold. Then $T\mathbb{C}_X$ has a complex subbundle $T_X^{0,1}$ defined as the $-i$ eigenspace of J. Equivalently, it is the *conjugate* subbundle of $T_X^{1,0} \subset T\mathbb{C}_X$. This is called the **antiholomorphic tangent space**.

(Skipping a little bit of stuff.)

1.3.5. *The Frobenius Theorem.* Let X be a smooth manifold. We understand a correspondence between vector fields on X and derivations of $C^\infty(X)$.[3] We define the **bracket** $[\chi, \psi]$ of two vector fields χ, ψ as the vector field corresponding to the derivation $\chi \circ \psi - \psi \circ \chi$.

Proposition 1.2. *In local coordinates x_i, write $\chi = \sum \chi_i \frac{\partial}{\partial x_i}$ and $\psi = \sum \psi_i \frac{\partial}{\partial x_i}$. Then*

$$[\chi, \psi] = \sum \left(\chi(\psi_i) - \psi(\chi_i)\right) \frac{\partial}{\partial x_i}.$$

[2] Of *real* vector bundles.

[3] In the $\to$ direction, a vector field $\chi : X \to T\mathbb{R}_X$ maps to $f \mapsto df(\chi)$.

2. Week 2 — Complex manifolds

2.1. **(8/29) Monday.** Today we want to introduce the Weierstrass theorems. Important notation: we regard

$$\mathbb{C}^n = \{(z,\omega) : z \in \mathbb{C}^{n-1}, \omega \in \mathbb{C}\}.$$

Definition 2.1. Let f be defined near 0 in $\mathbb{C}^n$. We say f is **regular** in ω at 0 if $f(0,\dots,0,\omega)$ is not identically zero near 0.

Definition 2.2. A **Weierstrass polynomial** (of degree d in ω) := an analytic function f defined in a neighborhood of 0 of the form

$$f(z,w) = \omega^d + a_1(z)\omega^{d-1} + \cdots + a_d(z)$$

Where $a_j(z)$ are holomorphic near zero in $\mathbb{C}^{n-1}$ and $a_j(0) = 0$.

Theorem 2.1 (Weierstrass Preparation Theorem)**.** *If f is holomorphic near the origin in $\mathbb{C}^n$ and regular (in ω at zero), then in a neighborhood of zero we may uniquely write $f = gh$, where*

- *g is a Weierstrass polynomial, and*
- *h is holomorphic with $h(0) \neq 0$.*

An important corollary is the following. Let $U \subset \mathbb{C}^n$ be open; say $Z \subset U$ is an **analytic hypersurface** if it is precisely the vanishing set of some $f \in \mathcal{O}(U)$. Then, locally, an analytic hypersurface is a branched cover of some open set in $\mathbb{C}^n$.

2.2. **(8/31) Wednesday.** First, we proved the Weierstrass division theorem.

Sheldon, after the proof: "Now the course starts." The rest of the day was setting up **complex manifolds and things on them**. It seems we are going to carefully review Voisin Ch. 2, today being the first thrust into that chapter. Thus, whatever we talk about will eventually be in the reading notes, so I will defer most of the details to the reading notes.

Definition 2.3. Recall that a C^k **manifold** is a topological space with an open cover $\{U_i\}$ together with homeomorphisms $\phi_i : U_i \to V_i \subseteq \mathbb{R}^n$ such that every $\phi_i \circ \phi_j^{-1}$ is C^k where it is defined. (Also, 2nd countable + Hausdorff...)

Definition 2.4. A **complex manifold** of dimension n is a C^k manifold of dimension $2n$ whose transition maps are holomorphic.

Important example: **Complex projective space**

$$\mathbb{P}^n := \mathbb{P}^n(\mathbb{C}) := \mathbb{C}^n - \{0\}/(z \sim \lambda z, \text{ for every } \lambda \in \mathbb{C}^\times).$$

The charts are $U_i := \{z \in \mathbb{P}^n : z_i \neq 0\}$ with $\phi_i : U_i \to \mathbb{C}^n$ given by

$$z \mapsto (z_0/z_i, \dots z_{i-1}/z_i, z_{i+1}/z_i, \dots z_n/z_i).$$

Definition 2.5. For X a smooth manifold, we say $f : X \to \mathbb{R}$ is C^k if $f \circ \phi_i^{-1}$ is C^k for all i. Likewise, if X is complex and $f : X \to \mathbb{C}$, we say f is holomorphic if $f \circ \phi_i^{-1}$ is holomorphic.

Then Sheldon wrote out the definition of a topological and C^k vector bundle. See Definition 1.3.

2.3. **(9/2) Friday.** Let X be a smooth n-manifold. We defined a **derivation at $x \in X$ of $C^\infty(X)_x$**. We used this to define the **real tangent bundle** $T\mathbb{R}_X$ whose fiber at x is the space of derivations at x. This allows us to define a **vector field**: it is a section of $T\mathbb{R}_X$. Also, a **real p-form** is a section of $\wedge^p(T\mathbb{R}_X^*)$.

Now suppose X is complex. We define $\mathcal{O}_{X,x}$ by taking $C^\infty(X)_x$ and replacing 'smooth' with 'holomorphic.' Then we use it to define the **holomorphic tangent bundle** T_X. This T_X defines a holomorphic vector n-bundle.

Now, three facts:

(1) T_X is a subbundle of $T\mathbb{C}_X$, the complexified real tangent bundle;
(2) There is a 'real part' map $T_X \to T\mathbb{R}_X$ which, on fibers, is an $\mathbb{R}$-linear isomorphism;
(3) There is a 'multiplication by i' endomorphism $T_X \to T_X$.

Considering (2) and (3), the complex structure has led to an endomorphism $J : T\mathbb{R}_X \to T\mathbb{R}_X$ such that $J^2 = -1$.

Definition 2.6. An **almost complex structure** on a smooth manifold X is an endomorphism $J : T\mathbb{R}_X \to T\mathbb{R}_X$ such that $J^2 = -1$.

The preceding discussion tells us that every complex manifold has an almost complex structure.

To conclude, we ask: **What is the $+i$ eigenspace of J on $T\mathbb{C}_X$?** The answer is T_X. (Next, we should ask what the $-i$ eigenspace is, which together with T_X will split $T\mathbb{C}_X$.)

3. Week 3 — IVT, IFT, and Integrability

3.1. **(9/9) Friday.** We opened with some setup to discuss almost complex structures. Basically Section 1.3.4.

Next, the inverse and implicit function theorems.

Theorem 3.1 (Inverse function theorem). *Let U, V be subsets of $\mathbb{C}^n$ and $\phi : U \to V$ a holomorphic map. If at $p \in U$ the differential $d\phi_p : T\mathbb{C}_{U,p} \to T\mathbb{C}_{V,f(p)}$ is an isomorphism,[4] then ϕ is biholomorphic[5] on some open neighborhood of p.*

Theorem 3.2 (Implicit function theorem). *See GH p. 18.*

Now, integrability. We understand vector fields as either (1) sections of $T\mathbb{R}_X$ or (2) derivations. Using (2), we define the **bracket** of two vector fields χ, ψ to be $\chi \circ \psi - \psi \circ \chi$. In the form (1), i.e. as a section, this looks locally how we expect it to.

Lemma 3.1. *Locally,*

$$[\chi = \sum \chi_i \frac{\partial}{\partial x_i}, \psi = \sum \psi_i \frac{\partial}{\partial x_i}] = \sum \left(\chi(\psi_i) - \psi(\chi_i)\right) \frac{\partial}{\partial x_i}.$$

On a complex manifold, the bracket of two holomorphic vector fields is again holomorphic. We express this by $[T_X, T_X] \subseteq T_X$. On an a.c. manifold, we have fundamental structure on T_X, and we can ask how the bracket respects it.

Theorem 3.3 (Newlander-Nirenberg). *Let (X, J) be an a.c. manifold. Then (X, J) is integrable $\iff [T_X^{0,1}, T_X^{0,1}] \subseteq T_X^{0,1}$.*

One should collapse in shock, dead, after reading this theorem. (Sheldon did not say that.)

[4] In other words, if the Jacobian with entries $\frac{\partial f_i}{\partial z_i}$ is nonsingular.

[5] An isomorphism in the category of complex manifolds.

4. Week 4 — Dolbeaut cohomology, Hermitian geometry

4.1. **(9/12) Monday.** We can do what we did last Friday but moreso. Let X be a real or complex manifold. A **rank k distribution** on X is a rank k sunbbundle of X's tangent bundle. (We can consider any type.) We say a distrubtion E is **integrable** if locally on X, we can find $\phi : U \to \mathbb{R}^{n-k}$ (or $\mathbb{C}$) such that $E = \ker \mathrm{d}\phi$.

Theorem 4.1 (Frobenius theorem)**.** *A distribution E is integrable $\iff$ $[E, E] \subseteq E$.*

Then we "defined" a complex submanifold.

Definition 4.1. Let M^n be a complex manifold. A **codim k submanifold** S is a closed subset satisfying either of the following.

(1) In charts U for M, $S \cap U = \{z : f_1(z) = \cdots = f_k(z) = 0 \text{ and } \det(\frac{\partial f_i}{\partial z_j}) \neq 0\}$.
(2) There are open $W \subset \mathbb{C}^{n-k}$ such that $S \cap U = \{(g_1\tilde{z}, \ldots, g_k\tilde{z}), \tilde{z}) : g_i \in \mathcal{O}(W), \tilde{z} = (z_{k+1}, \ldots, z_n\}$.

Then we changed subjects to forms.

Let X be a smooth manifold. Define $A^p(M)$ to be the vector space of smooth p-forms on X. Then we have the differential $d : A^p \to A^{p+1}$ satisfying $d^2 = 0$, thus we can define the **p-th de Rahm cohomology group** $\ker(d)/\mathrm{im}(d)$, and de Rahm's theorem says that these groups are just the singular cohomology groups.

If X is complex, we get more. As $T\mathbb{C}_X$ splits, we get $T\mathbb{C}_X^* = T_X^{1,0*} \oplus T_X^{0,1*}$. Now we make some definitions.

- $\Omega^{1,0} := T_X^{1,0*}$
- $\Omega^{0,1} := T_X^{0,1*}$
- $\Omega^{p,0} := \wedge^p \Omega^{1,0}$
- $\Omega^{0,p} := \wedge^p \Omega^{0,1}$
- $\Omega^{p,q} := \Omega^{p,0} \otimes \Omega^{0,p}$
- $A^{p,q}(X) := \{\ C^\infty \text{ sections of } \Omega^{p,q}\}$

These $A^{p,q}$ let us see more structure than A^k. Note that $A^k \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{p+q=k} A^{p,q}$, so we have $A^{p,q} \subseteq A^k \otimes \mathbb{C}$. Then $d(A^{p,q}) \subseteq A^{k+1} \otimes \mathbb{C}$. In fact, it is easy to see (check in local coordinates) that

$$d(A^{p,q}) \subseteq A^{p+1,q} \oplus A^{p,q+1}.$$

Thus we may write $d = \partial + \bar{\partial}$, which are defined to be the projection of d onto the $(p+1, q)$ and $(p, q+1)$ type parts, respectively. So we have a complex

$$0 \xrightarrow{\bar{\partial}} A^{0,1} \xrightarrow{\bar{\partial}} A^{0,2} \xrightarrow{\bar{\partial}} \cdots$$

Whose cohomology is the **Dolbeaut cohomology**. This Dolbeaut cohomology is *locally exact*; or more precisely, $\bar{\partial}$ is. This is the **$\bar{\partial}$-Poincaré lemma**, which was stated a week or two ago. We spent the rest of lecture beginning the proof of it.

4.2. **(9/14) Wednesday.** First, we finished the proof of the dee bar Poincare lemma.

Next up: we will see that holomorphic vector bundles have a "Dolbeaut cohomology." In the process of showing this, we will see why we use $\bar{\partial}$. It is roughly because the zero antiholomorphic part of holo. functions gives $\bar{\partial}$ nice properties that ∂ does not usually enjoy. (Why not use antiholomorphic bundles and ∂, then? Because there are no antiholomorphic bundles; the transitions would square to a holomorphic function.)

I'm going to do everything using only $\Omega^{0,q}$. Rick used $\Omega^{p,q}$.

Let $E \to X$ be a holomorphic vector k-bundle. Define

$A^{0,q}(E) := \{C^\infty$ **sections of** $E \otimes_{\mathbb{C}} \Omega_X^{0,q}\}$. Over a trivialized U, we can write a section $s \in A^{0,q}$ as $(\alpha_1, \ldots, \alpha_k)$ where α_i are smooth $(0,q)$-forms on U. Then we define

$$\bar{\partial}_U \alpha := (\bar{\partial}\alpha_1, \ldots, \bar{\partial}\alpha_k).$$

Lemma 4.1. *Let $E \to X, U \subset X$ be as above. If $\alpha \in A^{0,q}(E)$, then*

$$\bar{\partial}|_U \alpha_{U \cap V} = \bar{\partial}|_V \alpha_{U \cap V}.$$

Definition 4.2. Define

$$\bar{\partial}_E : A^{0,q}(E) \to A^{0,q+1}(E)$$

By the condition that $\bar{\partial}_E(\alpha)|_U = \bar{\partial}_U(\alpha|_U)$.

This definition is well-defined thanks to the preceding lemma. (The analogous lemma for ∂ is *not* true.) This has the same nice properties as (say) d, so we can form the **Dolbeaut complex of a holomorphic vector bundle** E $\cdots \to A^{0,q} \xrightarrow{\bar{\partial}} A^{0,q+1} \to \cdots$ and get the "Dolbeaut cohomology of a holomorphic vector bundle."

Change of topics: Hermitian geometry.

Let V be a complex vector space, $W_{\mathbb{R}}$ its real dual, $W_{\mathbb{C}} := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ its complex dual. The automorphism of V induces a splitting $W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$. We define

$$W^{1,1} = W^{0,1} \otimes W^{1,0} \subset \wedge^2 W_{\mathbb{C}}.$$

We look at $W_{\mathbb{R}}^{1,1} := W^{1,1} \cap \wedge^2 W_{\mathbb{R}}$. These forms have an important characterization.

Definition 4.3 (Hermitian form)**.** A **Hermitian form** on V is a pairing $h : V \times V \to \mathbb{C}$ satisfying

(1) $\mathbb{R}$-bilinearity,
(2) $\mathbb{C}$-linearity in the first variable,
(3) $h(z, w) = \overline{h(w, z)}$. (This implies $\mathbb{C}$-antilinearity in 2nd variable.)

Lemma 4.2. *There is a natural bijection*

$$\{\textit{Hermitian forms on } V\} \longleftrightarrow W_{\mathbb{R}}^{1,1}$$

Given by $h \mapsto -\mathrm{Im}\, h$. The inverse is $\omega \mapsto ((z, w) \mapsto \omega(z, Iw) - i\omega(z, w))$.

4.3. **(9/16) Friday.** More Hermitian stuff today. Say a Hermitian form is *positive* if it is positive-definite. This is also called a Hermitian metic. In case a manifold has a positive Hemitian form, it also has a Riemannian metric: $g = \mathrm{Re} h$.

Definition 4.4. A **Hermitian metric** on an a.c. manifold (X, I) is a Hermitian metric h_x on each $T\mathbb{R}_{X,x}$. We call it *smooth* if $h_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ is always smooth.

Definition 4.5 (Kähler form)**.** Let (M, I) be a.c. with a Hermitian metric h. Then $\omega := -\mathrm{Im}\, h \in \Omega_M^{1,1} \cap \Omega_{M,\mathbb{R}}^2$. We call ω the **Kähler form of** h or the **associated 1-1 form of** h.

Definition 4.6 (Kähler manifold)**.** Let (M, I) be a.c. with a Hermitian metric h. We say M is **Kähler** if I is integrable and the Kähler form of h is closed.

(Talked about examples: complex torii, $\mathbb{C}P^n$.)

(A couple more things)

5. Week 5 — Connections, Kähler manifolds

5.1. **(9/21) Wednesday.** Recall: IF M complex with a Hermitian metric h, GET

(1) The associated 1-1 form $\omega := -\operatorname{Im} h$,
(2) A Riemannian metric $g := \operatorname{Re} h$,
(3) An expression for the volume form: $\omega^n/n!$.

Now we do some geometry. Let X be smooth, $E \to X$ a smooth bundle with rank k.

Definition 5.1. A **connection** on E is an $\mathbb{R}$-linear map

$$\nabla : C^\infty(E) \to A^1(E) = \Omega_{X,\mathbb{R}} \otimes E = \operatorname{Hom}(T\mathbb{R}_X, E),$$

Such that $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$ for all $f : M \to \mathbb{R}$ and $\sigma \in C^\infty(E)$.

Locally, connections are easy to describe. Let U be trivialized, $e_i : U \to \mathbb{R}^k$ a local frame for E. Then $\nabla e_i = \sum \theta_{ij} \otimes e_j$, where the θ_{ij} are matrices of 1-forms.

Definition 5.2. Say ∇ is **compatible** with the metric g if for all $\phi, \psi \in A^0(T\mathbb{R}_M)$, we have

$$dg(\phi, \psi) = g(\phi, \nabla\psi) + g(\nabla\phi, \psi).$$

(Here, $\nabla\psi(t) =: \nabla_\psi(t)$ for a section ψ and vector field t.)

Theorem 5.1. *If (M, g) is Riemannian, then there exists a unique connection $\nabla : C^\infty(T\mathbb{R}_M) \to A^1(T\mathbb{R}_M)$ such that*

(1) ∇ is compatible with g, and
(2) ∇ is torsion-free.

*This is the **Levi-Cevita connection**.*

We can do this all again in Hermitian geometry. Let M be complex and $E \to X$ holomorphic. Now, a (complex) connection is a $\mathbb{C}$-linear map $A^0(E) \to A^1(E)$. Such connections can be compatible with the Hermitian metric h.

Theorem 5.2. *There exists a unique connection, the **Chern connection**, satisfying*

(1) ∇ satisfies $\nabla^{0,1} = \bar{\partial}_E$, and
(2) ∇ is compatible with h.

We wrap up with an important theorem. Let M be complex, $E = T_M$. We have a canonical isomorphism $T_M \cong T\mathbb{R}_M$. Let h be a Hermitian metric on T_M, then we have an associated Riemannian metric g on $T\mathbb{R}_M$.

Theorem 5.3. *TFAE:*

(1) The metric h is Kähler, i.e. the associated 1-1 form is closed;
*(2) I is **flat for the Levi-Cevita connection**, meaning*

$$\nabla_{LC}(I\chi) = I\nabla_{LC}(\chi), \quad \forall \chi \in A^0(T\mathbb{R}_X).$$

(3) The Chern connection and Levi-Cevita connection are identified under $T_X \cong T\mathbb{R}_X$.

5.2. **(9/23) Friday.**

6. Week 6 — New Kähler manifolds from old, blow-ups

6.1. **(9/26) Monday.** <u>**Case Study!**</u> Let's show $\mathbb{C}P^n$ is Kahler.

Recall that the **universal line bundle** S over $\mathbb{C}P^n$ is defined fiberwise as $\pi^{-1}(z) = \{\omega \in \mathbb{C}^{n+1} : \omega = \gamma z, \quad \gamma \in \mathbb{C}\}$. It naturally has a holomorphic structure. We denote by $\mathcal{O}_{\mathbb{P}^n}(1)$ the dual bundle S^*. Both S and $\mathcal{O}_{\mathbb{P}^n}(1)$ have natural Hermitian metrics h and h^*, coming from the restriction of the metric on $\mathbb{C}^{n+1}$.

Let $U_i := \{(z_0, \dots, z_n) : z_i \neq 0\}$. These trivialize S, S^*. Here's a frame for S: $\sigma_i(z) = (z_0, z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n)$. Thus, σ_i^* describes a frame for S^* on U_i. Unwrapping definitions (to-do: do so) gives us

$$\omega_i = \frac{1}{2\pi i} \partial\bar{\partial} h^*(\sigma_i^*),$$

Where ω_i is the real 1-1 form associated to h^*. We also have that $h^*(\sigma_i^*) = 1/\sigma_i$. We know what σ_i is, which gives us

$$h^*(\sigma_i^*) = \frac{1}{1 + \sum_{j \neq i} |z_j|^2}.$$

This puts ω_i into a form about which we can work with. In particular, we can show (we did) that ω is positive. Thus, the Hermitian inner product h^* is positive-definite, i.e. is a *metric*. So $\mathbb{C}P^n$ is Kahler.

Change of topic: New Kahler manifolds from old.

Proposition 6.1. *Submanifolds of Kahler manifolds are Kahler.*

Theorem 6.1 (Chow's Theorem)**.** *Any compact submanifold of* $\mathbb{C}P^n$ *is algebraic. (E.g., the zero sets of* n *homogeneous polynomials.)*

Definition 6.1. M complex, $E \to M$ holomorphic. We define $\mathbb{P}(E)$ to be the bundle with total space $(E - B)/\mathbb{C}^\times$. It has fibers $\cong \mathbb{C}P^{n-1}$. There is an evident projection map $\pi : \mathbb{P}(E) \to M$.

Proposition 6.2. *The total space* $\mathbb{P}(E)$ *is a complex manifold. Furthermore, if* M *is Kahler then* $\mathbb{P}(E)$ *is Kahler.*

6.2. **(9/28) Wednesday.** Sheldon opened with some comments about algebraic vs. complex geometry and the relationships between the objects each field considers.

Next: is $\mathbb{P}(E)$ a complex manifold? Yes: from the transition matrices $g^{UV} : U \cap V \to GL_r(\mathcal{O}_{U\cap V})$, we get transitions $[g^{UV}] \in PGL_r(\mathcal{O}_{U\cap V})$. Is it Kahler? Also yes.

Proposition 6.3. *Let* M *be compact and Kahler, and* $E \to M$ *a holomorphic vector bundle. Then* $\mathbb{P}(E)$ *is compact and Kahler.*

Proof. The proof is a 'relative' version of the proof that $\mathbb{C}P^n$ is Kahler. Recall that for $\mathbb{C}P^n$, we showed that the Hermitian inner product on $S^* = \mathcal{O}_{\mathbb{P}^n}(1)$ induced a positive (1,1)-form.

Now in the case of $\mathbb{P}(E)$, define S to be the subbundle of π^*E whose fiber at $[z]$ is $\{\omega : \omega = \lambda z\}$. So, it's a line bundle over $\mathbb{P}(E)$. We define $\mathcal{O}_{\mathbb{P}(E)}(1) := S^*$.

Let h be a Hermitian metric of E. This induces one on π^*E and hence on S, S^* by restriction. We consider the associated (1,1)-form ω_E. Fiberwise, ω_E is the form induced by the Fubini-Study metric on $\mathbb{P}(E_m)$, which itself comes from h_m on E_m. In particular, for $p \in \pi^{-1}(m)$, the form ω_E is positive on the subspace $T\mathbb{R}_{\pi^{-1}(m),p}$ of $T\mathbb{R}_{\mathbb{P}(E),p}$.

(FINISH) □

Next, Sheldon taught us some "folklore and culture." First: blow-ups.

<u>**Special case:**</u> Consider $\widetilde{\mathbb{C}^n} \subseteq \mathbb{C}^n \times \mathbb{P}^{n-1}$ defined as $\{(z, w) : z \in \omega\}$, or equivalently as $\{(z, w) : z_i w_j = z_j w_i \, \forall i, j\}$. Let $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ denote projection. See that

- π is an isomorphism away from the origin, and
- $\pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$.

We call $(\widetilde{\mathbb{C}^n}, \pi)$ the *blow-up of* $\mathbb{C}^n$ *at* 0. Griffiths and Harris say that $\widetilde{\mathbb{C}^n}$ is "the set of lines in $\mathbb{C}^n$ made disjoint." I'm not exactly feeling that, but they are wiser than I.

General case: Let X be complex, and let $Y \subseteq X$ be a complex codimension k submanifold. We may take charts U of X such that $U \cap Y$ is the zero set of holomorphic $f_1^U, \ldots, f_k^U \in \mathcal{O}(U)$ such that the df_i are independent. Define

$$\widetilde{U_Y} := \{(u, w) \in U \times \mathbb{P}^{k-1} : w_i f_j(u) = w_j f_i(u), \forall i, j \leq k\}.$$

Denote by $\tau : \widetilde{U_Y} \to U$ the evident projection. Over $U - Y$ it is an isomorphism, and over Y it has fiber $\mathbb{P}^{k-1}$. We want to glue these $\widetilde{U_Y}$ together.

Proposition 6.4. *Let U, V be open subsets of X in which Y is defined by f_i^U, f_i^V having independent differentials. Consider $\tau_U : \widetilde{U_Y} \to U$ and $\tau_V : \widetilde{V_Y} \to V$. There exists a natural isomorphism*

$$\phi_{UV} : \tau_U^{-1}(U \cap V) \cong \tau_V^{-1}(U \cap V)$$

Such that $\tau_U = \tau_V \circ \phi_{UV}$.

Definition 6.2 (Blow-up)**.** Let X be complex, Y a complex submanifold. We define the **blow-up** of X along Y to be the $\widetilde{U_Y}$ glued together along the identifications described in the above proposition. We write this manifold $\widetilde{X_Y}$. It has a canonical projection $\tau : \widetilde{X_Y} \to X$.

We worked through this, I won't write it out here. (Maybe I will?) But now we can say a few things.

Proposition 6.5. *If X is Kahler and $Y \subseteq X$ is a compact, complex submanifold, then $\widetilde{X_Y}$ is Kahler.*

In the above, compactness is necessary, and the converse is false. The proof predictably involves using a canonical line bundle over the blow-up.

Proposition 6.6 (Castelnuovo's contraction criterion)**.** *If X is a complex surface and $Y \subset X$ is a submanifold isomorphic to $\mathbb{P}^1$, then there exists a neighborhood of Y in X isomorphic to a neighborhood of $\mathbb{P}^1$ in $\widetilde{\mathbb{C}^2}$.*

6.3. **(9/30) Friday.** A lot of what I wrote for Wednesday was actually done today, or had its details worked out today.

For the last 4 minutes of class, we stated some basic definitions about sheaves. Nothing, really. We will probably start from scratch next Wednesday. (Monday is a problem session, no lecture.)

7. Week 7 — Sheaves

7.1. **(10/5) Wednesday.** Sheldon: A (pre)sheaf is a *global* object knowing *local* info.

Definition 7.1. Let X be a topological space, C a category. A C-valued **presheaf** is a contravariant functor $\mathsf{Op}(X) \to \mathsf{C}$, where $\mathsf{Op}(X)$ denotes the category of open sets of X whose morphisms are inclusions. Equivalently, it consists of the following data:

- For each open $U \subseteq X$, an object $\mathcal{F}(U) \in \mathsf{C}$;
- For each inclusion $V \subseteq U$, a morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$. This assignment must occur in such a way that $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$.

We frequently take $\mathsf{C} = \mathsf{Ab}$ or Ring, the categories of abelian groups or (unital, commutative) rings.

Examples:

(1) The sheaf C^∞ of smooth functions on a smooth manifold;
(2) The sheaf $\mathcal{O}$ of holomorphic functions on a complex manifold;
(3) Given a group or ring G, the *constant presheaf* assigning G to every nonempty set.

Definition 7.2. Let $\mathcal{F}$ be a presheaf.

(1) A *section* of $\mathcal{F}$ over U is an element $s \in \mathcal{F}(U)$.
(2) A *morphism of presheaves* $\mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction.

Definition 7.3. A **sheaf** is a presheaf $\mathcal{F}$ such that

- (Gluing) If $U = \bigcup V_i$ and $s_i \in \mathcal{F}(V_i)$ always agree on intersections after restriction, then there exists $s \in U$ restricting to each s_i.
- (Locality) If $s \in \mathcal{F}(U)$ and $s|_V = 0$ for all $V \subseteq U$, then $s = 0$.

Proposition 7.1 (Sheafification)**.** *If $\mathcal{F}$ is a presheaf, then there exists a unique sheaf $\mathcal{F}_f$ such that there exists a morphism $\phi : \mathcal{F} \to \mathcal{F}_f$ through which any other morphism $\mathcal{F} \to \mathcal{G}$ factors. See the diagram.*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \mathcal{F}_f \arrow[d, dashed, "\exists!"] \\ & \mathcal{G} \end{tikzcd}$$

7.2. **(8/7) Friday.** Examples of sheaves of $\mathcal{A}$-modules:

- $\mathcal{A}^n$, defined by $U \mapsto \bigoplus_{i=1}^n \mathcal{A}(U)$.
- If $\mathcal{A} = \mathbb{Z}$, then any sheaf of abelian groups is an $\mathcal{A}$-module.
- If $\mathcal{A} = C^0, C^\infty, \mathcal{O}$, then the sheaf of sections of a continuous/smooth/holomorphic vector bundle.

Definition 7.4 (Locally free sheaf)**.** Let $\mathcal{F}$ a sheaf of $\mathcal{A}$-modules. We call $\mathcal{F}$ a **locally free $\mathcal{A}$-module** if locally, we have an isomorphism of sheaves $\mathcal{F} \cong \mathcal{A}^n$ for some n.

Proposition 7.2 (Bundle $\iff$ locally free sheaf)**.** *Let $\mathcal{A}$ = one of $C^0, C^\infty, \mathcal{O}$. For E a cts/smooth/holomorphic vector bundle, let $\mathcal{E}$ denote its sheaf of sections. The mapping $E \mapsto \mathcal{E}$ defines a bijection between between vector bundles and locally free sheaves of $\mathcal{A}$-modules.*

Now, we define some sheaf terminology. In what follows, let $\mathcal{F}$ be a sheaf on X and $x \in X$.

Definition 7.5. Some terms.

- The **stalk at x** $\mathcal{F}_x$ is the colimit of $\mathcal{F}(U)$ as U ranges over open sets containing x.
- Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We denote by $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ the map induced by ϕ on stalks. (A representative $s \in \mathcal{F}_x$ is represented by a choice of element s_i in each U, and ϕ_x acts by $s_i \mapsto \phi(s_i)$.)
- We say **a morphism ϕ of sheaves is surjective/injective** if it is surjective/injective on every stalk.

Here's something important. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism. Consider the following presheaves:

$$U \mapsto \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)),$$
$$U \mapsto \mathrm{im}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)).$$

The first presheaf is a sheaf; we call it the **kernel sheaf** $\ker\phi$. The second presheaf is NOT a sheaf; we call its *sheafification* the **image sheaf** $\mathrm{im}\phi$. (To-do: Why? I want to believe it is because "being in the kernel" is a more stringent condition than "being in the image." What I'm really trying to say needs more thought.)

Proposition 7.3. *A morphism $\phi : \mathcal{F} \to \mathcal{G}$ is injective/surjective iff $\ker\phi \cong 0$/$\mathrm{im}\phi \cong \mathcal{G}$.*

8. Week 8 — Derived functors, abelian categories, sheaf cohomology

8.1. **(10/10) Monday.** First, we defined **subsheaves** and **complexes of sheaves**. Also, given a complex of sheaves $\mathcal{F}^\bullet$, we defined its **cohomology sheaf** $\mathcal{H}^i(\mathcal{F}^\bullet)$. It is the complex of sheaves whose components are the kernel/image quotient sheaves induced by $\mathcal{F}^\bullet$.

Definition 8.1 (Cokernel sheaf)**.** If $\varphi : \mathcal{F} \to \mathcal{G}$ denotes a morphism of sheaves, we define the **cokernel** of φ to be the sheaf $U \mapsto \mathcal{G}(U)/\mathrm{im}(\varphi(U))$. If $\ker\varphi = 0$ then we define the **quotient sheaf** $\mathcal{G}/\mathcal{F} := \mathrm{coker}\varphi$.

Definition 8.2. Let $\mathcal{F}$ be a sheaf. A **resolution of $\mathcal{F}$** is a complex of the form

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

Example 8.1 (De Rahm resolution)**.** Let X be a smooth manifold, $\mathbb{R}$ the constant sheaf on it. Consider the morphism $\mathbb{R} \to \Omega^0$ given by $t \mapsto (U \mapsto t)$. We now resolve $\mathbb{R}$, and call this the **de Rahm resolution:**

$$0 \to \mathbb{R} \to \Omega^0 \to \Omega^1 \to \dots$$

In fact, this resolution is *exact*. At $\mathbb{R}$ and Ω^0 this is obvious. At $\Omega^{>0}$, this is the Poincaré lemma. The reason that local lemma works is because of the sheaf axioms.

Example 8.2 (Dolbeaut resolution)**.** Let $E \to X$ be a holomorphic vector bundle. Recall (Week 4, 4.2) that we defined $\bar{\partial}_E : A^{0,p}(E) \to A^{0,p+1}(E)$ and defined Dolbeaut cohomology with it. Now consider $\bar{\partial}_E : A^{0,0}(E) \to A^{0,1}(E)$. I think we should think of $A^{0,0}(E)$ as the smooth sections of E. Such a section is locally described by some collection $(f_1, \dots, f_k)$ of smooth $\mathbb{C}$-valued functions. Thus, $s \in A^{0,0}(E)$ is in $\ker \bar{\partial}_E$ iff $\frac{\partial}{\partial \bar{z}} f_i = 0$ for every i. That is, iff s is a holomorphic section. Thus, we have a resolution:

$$0 \to \mathcal{O}(E) \to A^{0,0}(E) \to A^{0,1}(E) \to A^{0,2}(E) \to \dots$$

That right there is the **Dolbeaut complex of $E \to X$**. *This too is exact*, as we know that (i) $\bar{\partial}_E^2 = 0$ and (ii) there is the $\bar{\partial}$-Poincaré lemma.

Sheldon: Time to go off-book and give an informal overview of sheaf cohomology.

(I already know what he said, so I'm not going to write a lot.) We start with two facts: (I) there are SES'es of sheaves e.g. $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$; and (II) there is a *global sections functor* $\Gamma(X, -)$ that maps each sheaf $\mathcal{F}$ to $\mathcal{F}(X)$, and this functor is *left exact*, i.e. if you have that SES I just wrote, it is guaranteed that $0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H})$ is exact.

Now, we generally can't slap on $\to 0$ to that exact sequence and get an exact sequence. What *can* we slap on? The lesson from algebraic topology is that the original SES should induce a LES involving some kind of cohomology, *a cohomology associated to sheaves*. The result should look like this:

(TO-DO: make the diagram)

How to compute this "sheaf cohomology"? Well, we can consider a "resolution" $\mathcal{F}^\bullet$ of $\mathcal{F}$; if $\mathcal{F}^\bullet$ is "good," then

$$\text{sheaf cohomology of } \mathcal{F} \cong \text{cohomology of the complex } \Gamma(X, \mathcal{F}^\bullet).$$

Sheldon's "good" terminology is nonstandard, but there are many examples of "good" resolutions:

- *Fine* resolutions (they have patitions of unity)
- *Injective* resolutions (the ones we will focus on)
- *Flabby* resolutions, also called *flasque*

Example 8.3. The Dolbeaut and de Rahm resolutions are *fine* resolutions.

Example 8.4. Suppose X is a *locally contractible* space (e.g. a manifold). Then the constant sheaf $\mathbb{Z}$ has a good resolution by sheaves of singular chains. Hence,

sheaf cohomology of X with coefficients in the sheaf $\mathbb{Z} \cong$ singular cohomology of X with $\mathbb{Z}$ coefficients.

8.2. **(10/12) Wednesday.** Abelian category day.

Definition 8.3. A functor $\mathcal{F}$ between abelian categories is called **left-exact** if

$$0 \to A \to B \to C \to 0 \text{ is exact } \implies 0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \text{ is exact.}$$

Example 8.5. Here are two examples of left-exact functors:

- $\mathrm{Hom}(M, -) : \mathsf{Mod}_A \to \mathsf{Mod}_A$.
- For X a space and $\mathsf{Ab}(X)$:=the category of sheaves of abelian groups on X, the **global sections functor** $\Gamma(X, -) : \mathsf{Ab}(X) \to \mathsf{Ab}$ given by $\mathcal{F} \mapsto \mathcal{F}(X)$.

Definition 8.4. An object I of an abelian category is called **injective** if *every* map into I factors through *every* injection out of its domain. That is, the following diagram can be filled-in for any $X \hookrightarrow Y$ and $X \to I$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[dl, dashed] \\ I & \end{tikzcd}$$

Example 8.6. In Ab, the injective objects are the divisible groups.

Definition 8.5. An abelian category is said to **have enough injectives** if every object injects into an injective object.

Lemma 8.1. *If a categoy has enough injectives, then every object has a resolution by injective objects.*

Proof. Here's how you build the resolution for any object A:

(1) $0 \mapsto A \hookrightarrow I =: I^0$ (I^0 comes from having enough injectives)
(2) Can take cokernel (abelian cat) and then send cokernel into its injective (having enough injectives), thus get $I^0 \to \text{coker} \hookrightarrow I =: I^1$
(3) Compose: $0 \mapsto A \to I^0 \to I^1$
(4) Repeat

$\square$

(Proved some homological algebra stuff)

Definition 8.6. Suppose A has enough injectives and that $\mathcal{F} : \mathsf{A} \to \mathsf{B}$ is a left exact functor. For each $M \in \mathsf{A}$, choose an injective resolution $M^\bullet$. For every $i \geq 0$, define the i-th **right derived functor** of $\mathcal{F}$ as

$$R^i\mathcal{F}(M) := H^i(\mathcal{F}(M^\bullet)).$$

(The homological algebra I skipped proves that this definition is well-defined.)

Theorem 8.1. *The functors $R^i\mathcal{F}(-) : \mathsf{A} \to \mathsf{B}$ satisfy*

- *$R^0\mathcal{F}(M) = M$;*
- *For each SES $0 \to A \to B \to C \to 0$, there is a LES whose i-th 3-term row is $R^i\mathcal{F}(A) \to R^i\mathcal{F}(B) \to R^i\mathcal{F}(C)$. (TO-DO: just tex the damn diagram...)*
- *If I is injective, then $R^i\mathcal{F}(I)$ for every $i > 0$.*

8.3. **(10/14) Friday.** Sheldon went off-book again, and rushed so that we could cover what was necessary for the homework.

We started with the following.

Lemma 8.2. *Given $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ a left exact functor between abelian categories, if a sequence of functors $\{R^i\mathcal{F}\}_i$ satisfies the "right derived functor" properties, then on an object M, it must be given by the cohomology of the image of a resolution of M under $\mathcal{F}$:*

$$R^i\mathcal{F}(M) \cong H^i(\mathcal{F}(M^\bullet)).$$

Proof. Homological algebra. Sheldon's presentation of the proof was too fast for me. □

Injectives are not good for computations. Luckily, there are other classes of "good" resolutions that compute the derived functor, and these often come up in concrete situations.

Example 8.7 (De Rahm complex is fine)**.** For X a smooth manifold, let $A^i(X)$ denote the v.s. of smooth i-forms on X. Then $0 \to \mathbb{R} \to A^0 \to A^1 \to \dots$ is a resolution of the $\mathbb{R}$-constant sheaf. Each A^i is a sheaf of C^∞-modules, where C^∞ here is the sheaf of smooth functions. This sheaf $\mathbb{C}^\infty$ satisfies a certain "partition of unity" property, which is a consequence of the existence of smooth p.o.u.; this makes the A^i *fine sheaves.* Resolutions by fine sheaves are "good", so

$$H^k(X, \mathbb{R}) \cong \text{cohomology of } 0 \to A^0(X) \to A^1(X) \to \dots$$

The statement obtained by replacing $\mathbb{R}$ with $\mathbb{C}$ and real forms with complex forms is also true.

Example 8.8 (Dolbeaut complex is fine)**.** Let X be complex, $E \to X$ a holomorphic vector bundle, and $\mathcal{E}$ its corresponding locally free sheaf of $\mathcal{O}_X$-modules. If $A^{0,q}(E)$ denotes the sheaf of smooth sections of $\Omega^{0,q} \otimes E$, then we have the Dolbeaut resolution

$$0 \to \mathcal{E} \to A^{0,0}(E) \to A^{0,1}(E) \to \dots$$

These $A^{0,0}(E)$ are sheaves of C^∞-modules, so as before, they are fine, and this resolution computes $H^q(X, \mathcal{E})$.

The rest of the lecture was about Ĉech resolutions and cohomology. Sheldon has never spoken faster and my notes are irrecoverable. Just read the book.

9. Week 9 — Sheaf cohomology and complex geometry

9.1. **(10/23) Wednesday.** Today we elaborated on the notion of "good" resolutions.

Definition 9.1. $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ left exact, C with enough injectives. We say $M \in \mathsf{C}$ is **acyclic** (with respect to $\mathcal{F}$) if $R^i\mathcal{F}(M) = 0$ for all $i \geq 1$.

Proposition 9.1. *Suppose that $M^\bullet$ is an acyclic resolution of A. Then its cohomology computes the derived functor:*

$$R^i\mathcal{F}(A) \cong H^i(\mathcal{F}(M^\bullet)).$$

Proof. The idea is to take cokernels all over the resolution $0 \to A \to M^0 \to \dots$; this gives a bunch of SES's, and the isomorphisms can be inductively obtained from the resulting LES's. □

Example 9.1. Fine and flasque sheaves are acyclic with respect to the global sections functor $\Gamma(X, -)$.

Then we proved the following.

Proposition 9.2. *The category $\mathsf{Ab}(X)$ of sheaves of abelian groups has enough injectives.*

Proof. TO-DO: write this out. It's kind of interesting. □

Using this fact, we can construct an injective resolution for any sheaf, in fact canonically(?) This is the **Godemont resolution**. It is useless.

9.2. **(10/23) Friday.** Today and last Wednesday were Zoom classes. For some reason, today's class sounded like it was underwater. Nobody else had this problem. Today was spectral sequence day, though, so maybe I got lucky. Apparently, we covered something like pages 104-107 and 111-112 in Voisin; I'll just write up the takeaways.

In what follows, let $\mathcal{F}$ be a sheaf of abelian groups. Let $\mathcal{U} = (U_i)_i$ be a countable, ordered open covering of X.

Definition 9.2 (Pushforward, restriction)**.** For finite $I \subset \mathbb{N}$, write $U_I := \cap_I U_i$ and $j_I : U_I \hookrightarrow X$ for its inclusion. Let $j : V \hookrightarrow X$ be an inclusion of any open set. We define some sheaves now.

- Define the sheaf $j_*\mathcal{F}$ on X by $U \mapsto \mathcal{F}(U \cap j(V))$. (**Pushforward**)
- Define the sheaf $j^*\mathcal{F}$ on V by $W \mapsto \mathcal{F}(j(W))$. Also denoted $\mathcal{F}|_V$. (**Restriction**)
- Define the sheaf $\mathcal{F}_I$ on X as $j_I^*(\mathcal{F}|_{U_I})$. That is, the pushforward of the restriction.

Definition 9.3 (Cech resolution)**.** Define

$$\mathcal{F}^k := \bigoplus_{|I|=k+1} \mathcal{F}_I.$$

Also, define

$$d : \mathcal{F}^k \to \mathcal{F}^{k+1}$$

In the following way. Let $\sigma = \{\sigma_I \in \mathcal{F}_I\}_{|I|=k+1} \in \mathcal{F}^k(U)$. Now, for every $k+2$ subset $J \subset \mathbb{N}$, define

$$(d\sigma)_J := \sum_i (-1)^i (\sigma_{j_0 \dots \hat{j_i} \dots j_{k+1}})|_{U \cap U_J}$$

One checks that $d^2 = 0$. In fact, the following is a resolution, the **Cech resolution**:

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

Definition 9.4 (Cech cohomology)**.** Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathcal{U} = \{U_i\}$ be a countable, ordered open cover. Define the q-th **de Rahm cohomology** group $\check{H}^q(\mathcal{U}, \mathcal{F})$ of X (with respect to $\mathcal{U}$) to be the q-th cohomology group of the complex of global sections of the de Rahm resolution of $\mathcal{F}$. That is, it is the cohomology of the complex with objects

$$C^q(\mathcal{U}, \mathcal{F}) := \Gamma(U_I, \mathcal{F}^q) = \bigoplus_{|I|=q+1} \mathcal{F}(U_I).$$

Definition 9.5 (Brief generalities on double complexes)**.** A **double complex** $(K^{p,q}, D_1 : K^{p,q} \to K^{p+1,q}, D_2 : K^{p,q} \to Kp, q+1)$ in an abelian category is what you think it is. Notably, we demand $D_1^2 = D_2^2 = 0$ and $D_1 \circ D_2 = D_2 \circ D_1$. If a double complex $K^{p,q}$ is such that $p >> 0$ or $q >> 0$ implies $K^{p,q} = 0$, then the **total complex** K^n is the complex described as follows:

$$K^n := \bigoplus_{p+q=n} K^{p,q}, \quad D = D_1 + (-1)^p D_2$$

The double complex stuff is only necessary for the proof of the next proposition, which is in Voisin.

Proposition 9.3. *If the open cover $\mathcal{U}$ is such that $H^q(U_I, \mathcal{F}) = 0$ for all $q > 0$ and I, then*

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \cong H^q(X, \mathcal{F}), \quad \forall q \geq 0.$$

Theorem 9.1. *Suppose $\mathcal{F}$ is a sheaf over X, $\mathcal{U}$ is a countable open cover, $I^\bullet$ is an injective resolution of $\mathcal{F}$. By the universal property of injective resolutions, there is a canonical morphism of complexes $\mathcal{F}^\bullet \to I^\bullet$. Thus, there is a canonical morphism $\check{H}^q(\mathcal{U}, \mathcal{F}) \to H^q(X, \mathcal{F})$. If X is seperable, these assemble into an isomorphism in the colimit (direct limit):*

$$\lim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{F}) \cong H^q(X, \mathcal{F}).$$

10. Week 10 - Hodge star, adjoints, and elliptic operators

10.1. **(10/24) Monday.** We are returning to differential forms. First, we recall some general theory.

Proposition 10.1 (Basic structure induced by metric)**.** *Let V be an oriented, f.d., real vector space. For a choice of metric $g : V \times V \to \mathbb{R}$, we get:*

- *A canonical isomorphism $\psi_g : V \to V^*$ given by $v \mapsto g(v, -)$;*
- *For each oriented, orthonormal basis $\{e_i\}$ of V, the same sort of basis for V^*, namely $\{\psi_g(e_i)\}$;*
- *An induced metric on V^* given by $(\eta, \theta) \mapsto \eta(\psi_g^{-1}\theta)$;*
- *(Example computation: $(e_i^*, e_j^*) = e_i^*(e_j) = \delta_{ij}$.)*
- *In fact, an induced metric on the exterior powers $\wedge^k V^*$.*

Example 10.1. If (X, g) is a Riemannian n-manifold, then it has a canonical metric $(-,-)_x$ on each fiber $\wedge^k T^*_{X,x}$. At a point x, if e_i are an orthonormal basis for $T_{X,x}$, then the set of all $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ form an orthonormal basis for $\Omega^k_{X,x}$. Also, there is an associated **volume form** $Vol \in \Omega^n_X$. At x, it is (up to a multiple?) given by $e_1^* \wedge \cdots \wedge e_n^*$.

Definition 10.1. Consider the vector space $A^k(X)$ of smooth k-forms on X. The **L_2-metric on $A^k(X)$** is given by

$$(\omega, \eta)_{L^2} := \int_X (\omega, \eta) Vol,$$

Where (ω, η) is the function $x \mapsto (\omega_x, \eta_x)_x$.

Now we're going to make a series of easy identifications whose composite is very important. Let V be an oriented, n-dimensional vector space with a metric.

- There is a canonical $\wedge^n V \cong \mathbb{R}$; it normalizes the volume form.
- There is a canonical $\wedge^k V \cong \mathrm{Hom}(\wedge^k V, \mathbb{R})$; this is just the canonical identification induced by the metric on V.
- There is a canonical $\wedge^k V \cong \mathrm{Hom}(\wedge^{n-k} V, \wedge^n V)$; this is the right exterior product.

Definition 10.2 (Hodge star operator)**.** Let (X, g) be a Riemannian manifold. As in the example, at $x \in X$, the smooth forms $\wedge^k T^*_{X,x} = \Omega^k_{X,x}$ come with a metric. Using the above identifications, we form the operator $(*)_x$ as follows:

$$\wedge^k T^*_{X,x} \cong \mathrm{Hom}(\wedge^k T^*_{X,x}, \mathbb{R}) \cong \mathrm{Hom}(\wedge^K T^*_{X,x}, \wedge^n T^*_{X,x}) \cong \wedge^{n-k} T^*_{X,x}.$$

This varies smoothly in x (since g is smooth.) Thus we get an isomorphism of bundles

$$* : \Omega^k_{X,\mathbb{R}} \cong \Omega^{n-k}_{X,\mathbb{R}}.$$

This is the **Hodge star operator**. We may also consider it as the induced map on sections (i.e., smooth forms) $* : A^k(X) \to A^{n-k}(X)$.

Proposition 10.2 (Basic properties of $*$)**.** *On $A^k(X)$, the Hodge star operator satisfies $*^2 = (-1)^{k(n-k}$. Also,*

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta.$$

Proof. I think that, once one has thought long enough about the last identification in the definition of $*$, that it becomes clear that $\alpha_x \wedge *\beta_x = (\alpha_x, \beta_x)_x Vol_x$; hence the second statement follows.

Starting from that same equality, see that

$$(\alpha_x, \beta_x)_x Vol_x = (*\alpha_x, *\beta_x)_x Vol_x = *\beta_x \wedge (**\alpha_x) = (-1)^{k(n-k)}(**\alpha_x) \wedge *\beta_x$$

∎

The differential $d : A^k(X) \to A^{k+1}(X)$ is a linear operator. In the L_2-metric, d has an adjoint d^*, i.e. a map such that $(d\alpha, \beta)_{L^2} = (\alpha, d^*\beta)_{L^2}$.

Definition 10.3. Here are some definitions.

- d^* : $A^{k+1}(X) \to A^k(X)$ is the composite $(-1)^k(*)^{-1}d(*)$. Using Stokes', one verifies that this is adjoint to d.
- Δ_d := the **d-Laplacian** $:= dd^* + d^*d : A^k(X) \to A^k(X)$.
- Say $\omega \in A^k(X)$ is **Harmonic** if $\Delta_d(\omega) = 0$.
- $\mathcal{H}^k(X)$:= {harmonic k-forms} $\subseteq A^k(X)$.

Remark 10.1. See that $(\omega, \Delta_d\omega) = (d\omega, d\omega) + (d^*\omega, d^*\omega)$. With that, see that ω is harmonic iff $d\omega = d^*\omega = 0$. (In the book, Corollary 5.13; there's one step I don't understand.)

Theorem 10.1. *The natural map $\mathcal{H}^k(X) \to H^k_{dR}(X) \to H^k(X, \mathbb{R})$ is an isomorphism.*

10.2. **(10/26) Wednesday.** First, we did a simple, concrete computation. Namely, consider $\mathbb{R}^2$ with the Euclidean metric. It has an orthonormal frame of 1-forms $\{dx, dy\}$, and a canonical top-form $vol := dx \wedge dy$. Then we worked out

- $*1 = dx \wedge dy$,
- $*dx = dy$,
- $*dy = -dx$,
- $*dx \wedge dy = 1$,
- $d^*1 = 0$,
- $d^*(f dx) = -f_x$,
- $d^*(f dy) = f_y$,
- $d^*(h dx \wedge dy) = -h_x dy + h_y dx$,
- $\Delta_d f = -(f_{xx} + f_{yy})$,
- $\Delta_d(f dx) = -(f_{xx} + f_{yy})dx$.

Suppose X is complex. Since $* : A^k \to A^k$ is $\mathbb{R}$-linear, we may extend it to a linear map on $A^k(X) \otimes \mathbb{C}$. Thus, $*dz = dy - idx = -idz$.

Definition 10.4. Let (X, g) be Riemannian, and take a Hermitian metric on X (i.e., a smooth choice of positive Hermitian form on fibers of $T\mathbb{R}_X$; see Week 4) which extends g. This induces a Hermitian metric on complex-valued forms $\Omega^k_{X,\mathbb{C}}$. Then we may define the L_2-metric on complex-valued forms:

$$(\omega, \eta)_{L^2} := \int_X (\omega, \eta)_x vol_x.$$

Definition 10.5. Recall that we decomposed $d = \partial + \bar{\partial}$. We define

- $\partial^* := (*)^{-1}\bar{\partial}(*)$,
- $\bar{\partial}^* := (*)^{-1}\partial(*)$,
- $\Delta_\partial := \partial\partial^* + \partial^*\partial$,
- $\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$.

Proposition 10.3. *$\bar{\partial}^*$ is adjoint to $\bar{\partial}$ and similarly for ∂^*, in the L^2-metric for complex forms.*

Now, we want to extend all this harmonic theory to work for (sections of) holomorphic bundles $E \to X$. Recall that holomorphic bundles had something which general smooth bundles did not: the $\bar{\partial}$ operator on sections $A^{p,q}(E) \to A^{p,q+1}(E)$. This was defined so that over a trivialized U, given a holomorphic frame $\{e_i\}$ for E, for any section $\sigma := \sum \omega_I \otimes e_I \in A^{p,q}(E)$ one has $\bar{\partial}\sigma = \sum(\bar{\partial}\omega_I) \otimes e_I$.

We once and for all choose a Hermitian metric on X and E. This induces a metric on $A^{p,q}(E)$. *What is the Hodge * in this context?* Let's do this step-by-step.

- Start with $\Omega^{0,p}_X \otimes E$.

- It has a natural pairing with $\Omega_X^{n,n-p} \otimes E^*$.
- The result: a perfect pairing to $\Omega_X^{n,n}$. Note that $\Omega_X^{n,n}$ is isomorphic to the trivial $X \times \mathbb{C}$.
- In other words, we get a $\mathbb{C}$-**anti**linear

$$\Omega_X^{n,n-p} \otimes E^* \cong \mathrm{Hom}(\Omega_X^{0,p} \otimes E, \mathbb{C}).$$

- The Riemannian metric gives a natural identification $\mathrm{Hom}(\Omega_X^{p,0} \otimes E, \mathbb{C}) \cong \Omega_X^{0,p} \otimes E$.
- All together, we get a $\mathbb{C}$-**anti**linear isomorphism, the **Hodge * operator for E**

$$*_E : \Omega_X^{0,p} \otimes E \to \Omega_X^{n,n-p} \otimes E^*.$$

Notation: We write $K_X := \Omega_X^{n,0}$, called the **canonical bundle on X**.

Definition 10.6. Now we can define $\bar{\partial}_E^* := (*)^{-1}\bar{\partial}_{K_X \otimes E^*}(*)$. This is an operator $A^{0,q}(E) \to A^{0,q-1}(E)$.

Proposition 10.4. *$\bar{\partial}_E^*$ is formally adjoint to $\bar{\partial}_E$.*

10.3. **(10/28) Friday.** For my own sake, let me recap.

- Given X smooth, compact, oriented, with a smooth metric $g \rightsquigarrow$ The induced metric on fibers of $\Omega_{X,\mathbb{R}}^k$ gives rise to the operator (isomorphism) $* : A^k(X) \to A^{n-k}(X)$. It also gives rise to the L^2-metric on smooth k-forms $A^k(X)$. In this metric, $d : A^k(X) \to A^{k+1}(X)$ has a formal adjoint $d^* = (*)^{-1}d(*)$. Then we can define $\Delta_d = dd^* + d^*d$.
- Given X complex with a Hermitian metric $\rightsquigarrow$ finer structure on forms, i.e. (p,q)-forms $A^{p,q}(X)$, onto which we extend $d : A^{p,q}(X) \to A^{p+1,q} \oplus A^{p,q+1}$. Then we get $\partial, \bar{\partial}$. Since $A^{p,q}$ naturally includes into $A^k \otimes \mathbb{C}$, and because $* : A^k \to A^k$ is $\mathbb{R}$-linear, we get an extension $* : A^{p,q} \to A^{n-p,n-q}$. Then we can do the same thing to get adjoints/Laplacians: we define $\partial^* := (*)^{-1}\partial(*)$, $\bar{\partial}^* := (*)^{-1}\bar{\partial}(*)$, and similarly $\Delta_\partial, \Delta_{\bar{\partial}}$.
- Given a holomorphic bundle $E \to X$ and a Hermitian metric on E and $X \rightsquigarrow$ metric on $\Omega_X^{p,q} \otimes E$ and canonical identifications which give us a *complex antilinear* isomorphism $*_E : \Omega_X^{0,q} \otimes E \to \Omega_X^{n,n-q} \otimes E^*$. Now, we define $\bar{\partial}_E^* := (*)^{-1}\bar{\partial}_{K_X \otimes E^*}(*)$, where $K_X := \Omega_X^{n,0}$. We can define an L^2-metric on $A^{0,q}(E)$, and in this metric $\bar{\partial}_E^*, \bar{\partial}_E$ are adjoint. And finally, we define $\Delta_E := \bar{\partial}_E\bar{\partial}_E^* + \bar{\partial}_E^*\bar{\partial}_E$.

Remark 10.2. Here are two observations:

- All the above depends on a choice of metric. There is not necessarily a "right" choice.
- Take $E = \Omega_X^{p,0}$; then $A^{0,q}(E) \cong A^{p,q}(X)$.

(Q) How do our metric-dependent notions relate via this isomorphism? (E.g., how can we compare $\bar{\partial}_E$ to $\bar{\partial}$?) (A) TO-DO: figure this out precisely.

These things look similar. It is valuable to think about them as part of the general theory of *differential operators.*

Definition 10.7. Let $E, F \to M$ be smooth vector bundles over a manifold M, rank p and q resp. Let $P : C^\infty(E) \to C^\infty(F)$ be a (real or complex linear) morphism of sheaves. We say P is a **differential operator** of order k if:

Over trivialized U with coordinates x_i, $P(\alpha_1, \ldots, \alpha_p)$ is of the form $(\beta_1, \ldots, \beta_q)$ where

$$\beta_i = \sum_{I,j} P_{I,i,j} \frac{\partial \alpha_j}{\partial x_I},$$

Such that the $P \in C^\infty(U)$, vanish when $|I| > k$, and are nonzero for at least one I such that $|I| = k$.

Example 10.2. The classical Laplacian Δ_d is a map $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ acting by $f \mapsto (\frac{\partial^2 f}{\partial x_i^2})$. It is a second-order differential operator. This description is still true for trivial bundles over (Riemannian?) manifolds.

Let (M, g) be a Riemannian manifold and let $\pi : E \to M$ be a vector bundle. Let $\Gamma(E)$ denote the sections of E, regarded as a vector space. Let $D : \Gamma(E) \to \Gamma(E)$ be a linear map, and suppose that for every $p \in M$ we can find local coordinates $\varphi : U \subseteq M \to \mathbb{R}^n$ and a local trivialization $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ around p, such that for any local section $\mu : U \subseteq M \to \pi^{-1}(U)$ we have

$$\begin{aligned} D\mu &= \sum_{|\alpha| \le m} a_\alpha(p) \partial_x^\alpha (\pi_{\mathbb{R}^k} \circ \phi \circ \mu) \\ &= \sum_{|\alpha| \le m} a_\alpha(p) \partial_x^\alpha (\widehat{\mu}) \end{aligned}$$

then D is called a (order m) ***differential operator*** on sections of E, or just a ***differential operator*** on E. The main idea here is that, after choosing local coordinates and local trivializations, sections of E just look like smooth maps $\mathbb{R}^n \to \mathbb{R}^k$, and D just acts like a Euclidean differential operator.

Definition 10.8. WHAT IS THE SYMBOL OF A DIFFERNETIAL OPERATOR?
Following the above notation, the ***principal symbol*** of D is the map $\sigma_m(D) : T^*M \to End(E)$ given by

$$\sigma_m(D)(p, \omega_p) = \sum_{|\alpha| = m} a_\alpha(p) \omega^\alpha$$

where $\omega = \omega_1 dx^1 + \cdots \omega_n dx^n$ in local coordinates and $\omega^\alpha = w_1^{\alpha_1} \cdots \omega_n^{\alpha_n} \in C^\infty(M)$.

The main idea here is that we're just taking just top-order term of D and replacing the partial derivatives with cotangent variables. In particular, notice that although differential operators do not commute, their principal symbols do commute: $\sigma_{m_1}(D_1) \circ \sigma_{m_2}(D_2) = \sigma_{m_2}(D_2) \circ \sigma_{m_1}(D_1)$.

Definition 10.9. We say that a differential operator P is **elliptic** if for all $m \in M$ and $\alpha \in \Omega_{M,m}$, the induced $\sigma_{P,m} : E_m \to F_m$ is injective.

Elliptic operators have good properties, and we will do well to notice that the operators we care about are elliptic.

Proposition 10.5. *Elliptic operators have adjoints. (In the L^2 norm?)*

Theorem 10.2 (Fundamental theorem)**.** *If $P : C^\infty(E) \to C^\infty(F)$ is elliptic, then*

(1) The kernel of P is finite-dimensional;
(2) The image of P is finit-codimensional; and
(3) $C^\infty(E)$ splits orthonormally as $\ker(P) \oplus P^*(C^\infty(F))$.

Theorem 10.3. *If (X, g) is Riemannian and compact, then $\Delta_d, \Delta_\partial, \Delta_{\bar{\partial}, \Delta_E}$ are elliptic.*

11. Week 11 - Applications to harmonic forms, duality

11.1. **(11/2) Wednesday.** Let $\mathcal{H}^k_d(X) \subseteq A^k(X)$ denote the vector subspace of Δ_d-closed k-forms. As the kernel of Δ_d, which we know to be elliptic, Theorem 10.2 gives us the following.

Proposition 11.1. *If (X, g) is a compact Riemannian manifold, then we get an orthogonal decomposition $A^k(X) \cong \mathcal{H}^k(X) \oplus \Delta_d(A^k(X))$.*

With the proposition in mind, we think about the map $\mathcal{H}^k(X) \to H^k(X, \mathbb{R})$ given by $t \mapsto [t]$. Let $\beta \in A^k(X)$ be given; by the prop., we may write $\beta = h + \alpha = h + dd^*\gamma + d^*d\gamma$ for some $h \in \mathcal{H}^k$ and $\gamma \in A^k$. Suppose β is closed. This implies $dd^*d\gamma = 0$. So $d^*d\gamma$ is d-closed. It is also obviously d^*-closed; since our decomposition 11.1 was orthogonal, this demands $d^*d\gamma = 0$. Thus $\beta = h$ modulo an exact form. We conclude that $\mathcal{H}^k(X) \to H^k(X, \mathbb{R})$ is surjective.

By a shorter argument, one can also show the map is injective. We obtain the following.

Proposition 11.2. *The natural map $\mathcal{H}^k(X) \to H^k(X, \mathbb{R})$ is an isomorphism.*

In a totally analogous fashion, one obtains the following.

Proposition 11.3. *Let $\mathcal{H}^{p,q}(E)$ denote the space of E-valued (p, q)-forms killed by Δ_E. Recall that the Dolbeaut resolution of $\mathcal{E}$ (the sheaf of holomorphic sections of E) is fine, therefore $H^{p,q}(X; \mathcal{E})$ is identifiable with closed forms modulo exact ones (in E). Thus, as before, there is a comparison $\mathcal{H}^{0,q}(X) \to H^q(X, \mathcal{E})$. The assertion is that this map is an isomorphism.*

Corollary 11.1. *If X is compact, then $H^q(X; E)$ is finite-dimensional.*

(Note: the proof of the above uses the preceding proposition. This is "not easy." The analogous proposition for de Rham cohomology, i.e. the real case, is elementary.)

11.2. **(11/4) Friday.**

12. Week 12 - Kähler identities, Lefschetz decomposition

12.1. **(11/9) Wednesday.** (This is an amalgam of today, last Monday, and last Friday.) Now the plan is to see how the various operators we have defined ($\partial, \bar{\partial}$, adjoints, Laplacians) behave in the presence of a Kähler metric. I think, in some form, the Kähler condition is secretly saying "force this structure to be coherent."

Let (X, ω) be a Kähler manifold. The **Lefschetz operator** $L : A^k(X) \to A^{k+2}(X)$ is given by $- \wedge \omega$. It is order zero as an operator. Recall that a Kähler form ω induces a metric $(-,-)_x$ on each fiber $\Omega^k_{X,x}$. In this metric, L has an adjoint $\Lambda \colon A^k \to A^{k-2}$. It is given by $*^{-1}L* = (-1)^k * L*$.

Proposition 12.1 (Kähler identities)**.** *We have the identities* $[\Lambda, \bar{\partial}] = -i\partial^*$ *and* $[\Lambda, \partial] = i\bar{\partial}^*$.

Proof. First Two reductions. First, we need only prove these identities on $\mathbb{C}^n$ with the standard Euclidean metric; this is because the identities are first-order equations, and some very important principle says that if it holds on $\mathbb{C}^n$, then it works for any Kähler X. Second, we need only prove one identity; the other would follow by conjugacy.

The proof in Voisin uses the symbol. I do not understand that. The proof is Griffiths-Harris is more elementary, but a bit long. Let me summarize. As in the above, we may take $X = \mathbb{C}^n$ with the standard Euclidean metric.

(1) Define six "elementary operators" like so. Let (1) $e_k : A^{p,q}(X) \to A^{p+1,q}(X)$ be given by $\phi \mapsto dz_k \wedge \phi$. Let (2) $\bar{e_k}$ be given by wedging with $d\bar{z}_k$. Let (3,4) $i_k, \bar{i}_k$ denote their adjoints. Define (5) $\partial_k : A^{p,q}(X) \to A^{p,q}(X)$ by $\sum \phi_{IJ} dz_I \wedge d\bar{z}_J \mapsto \sum \frac{\partial \phi_{IJ}}{\partial z_k} dz_I \wedge d\bar{z}_J$. Define (6) $\bar{\partial}_k$ likewise ($\bar{z}_k$ derivative.)
(2) Prove various elementary facts about these elementary operators. E.g., ∂_k and $-\bar{\partial}_k$ are adjoint.
(3) Since we are using the Euclidean metric, we can write $L = (i/2)\sum e_k \bar{e}_k$ and $\Lambda = -(i/2)\sum \bar{i}_k i_k$.
(4) Now we can express $\Lambda\partial - \partial\Lambda$ in terms of our elementary operators and simplify; we find that it equals $i\bar{\partial}^*$.

This proves the first identity. The second can be proven similarly. It also follows by conjugacy. $\square$

So, the Kähler identites are more-or-less trivial on $\mathbb{C}^n$ with the usual Kähler metric. Then the Kähler condition necessitates that the identity holds on a general Kähler manifold. We get the following.

Corollary 12.1. *If* (X, ω) *is Kähler, then* $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.

Proof. Starting from Δ_d, one expands and applies the Kähler identities to get $2\Delta_\partial$ and $2\Delta_{\bar{\partial}}$. $\square$

Corollary 12.2 (The Hodge decomposition)**.** *Suppose that X is Kähler. Let $H^{p,q}$ and $\mathcal{H}^{p,q}$ denote the closed modulo exact (p,q)-forms and Δ_d-harmonic (p,q)-forms, respectively. Let us make some observations.*

- *Since $\Delta_d = 2\Delta_\partial$, that Δ_∂ is bihomogenous implies Δ_d is. Then by bihomogeneity, if α is a harmonic k-form written $\sum \alpha^{p,q}$, that $\Delta_d \alpha = 0$ implies $\Delta_d \alpha^{p,q} = 0$. This gives us a decomposition $\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$.*
- *Now we want to relate $\mathcal{H}^{p,q}(X)$ to $H^{p,q}$. We did this last week but I skipped details. The important point is that since Δ_d is bihomogeneous (explained above) and elliptic, we get a decomposition $A^{p,q} = \mathcal{H}^{p,q} \oplus \Delta_d(A^{p,q})$. By an argument identical to the one given for Proposition 11.2, one finds that the obvious comparison is an isomorphism:*

$$\mathcal{H}^{p,q} \cong H^{p,q}.$$

(TO-DO: What do they mean in GH by $dd^*G(\eta)$, p. 116?)

- *Remark: We can obtain the above fact from a more general angle. We work w.r.t. any holomorphic $E \to X$. Choosing a hermitian metric gives rise to a Laplacian $\Delta_E : A^{p,q}(E) \to A^{p,q}(E)$. We can identify Δ_E-harmonic forms $\mathcal{H}^{p,q}_E(X)$ with $\bar{\partial}_E$-closed (p,q)-forms modulo exact ones. If $E = \Omega^{p,0}_X$, then this induces an identification $\mathcal{H}^{p,q} \cong H^{p,q}_{\bar{\partial}}$. If the metric is Kähler, we additionally have $H^{p,q}_{\bar{\partial}} = H^{p,q}$, in which case*
$$\mathcal{H}^{p,q} \cong H^{p,q}.$$

We put this together into a decomposition
$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

Corollary 12.3. *Complex conjugation acts naturally on $H^{p+q}(X, \mathbb{C}) = H^{p+q}(X, \mathbb{R}) \otimes \mathbb{C}$. We have $\overline{H^{p,q}} = H^{q,p}$.*

That's the Hodge decomposition. Essentially, the Kahler condition let us relate singular cohomology to harmonic forms, and in this setting we could prove interesting things.

...

Let X be compact, Kahler n-manifold. We think about the Lefschetz operator and its adjoint
$$L := \omega \wedge -, \qquad \Lambda = *^{-1} L * .$$

Proposition 12.2. *Let $u \in A^k(X)$ be a k-form. We have the commutation relations*
$$[L, \Lambda]u = (k-n)u \quad \textit{and} \quad [L^r, \Lambda] = (r(k-n) + r(r-1))L^{r-1}.$$

So the commutator acts as multiplication by $k - n$ on $A^k(X)$. Now denote $h := [L, \Lambda]$; one checks that $[h, L] = -2L$ and $[h, \Lambda] = 2\Lambda$. Briefly, this gives us a "Lie algebra representation-theoretic" way to think about $A^k(X)$. Meanwhile,

13. Week 13 - Algebraic geometry?

13.1. **(11/14) Monday.**

13.2. **(11/16) Wednesday.**

Definition 13.1. Let $U \subseteq \mathbb{C}^n$ be open. A subset $V \subseteq U$ is called an **analytic variety** if every $p \in U$ has a neighborhood W such that $W \cap V$ is the common zero locus of finitely many holomorphic functions $f_1, \ldots, f_k$ on W. If $k = 1$, it is called an **analytic hypersurface**.

13.3. **(11/18) Friday.**