THE DERIVED FUNCTOR APPROACH TO SHEAF COHOMOLOGY

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ABSTRACT. Sheaves and their cohomology have transformed the study of complex and algebraic geometry over the last eighty years. The classical formulation of sheaf cohomology proceeds by combinatorial constructions on open covers and is now called Čech cohomology. The modern formulation is more algebraic and categorical, and uses the powerful but far-abstracted *derived functor*. This paper is a technical exposition of the modern approach, with modest prerequisites.

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This paper is a development of sheaf cohomology using derived functors. The reader should know basic category theory and homological algebra. No background is assumed about sheaves, abelian categories, or derived functors. Except where stated, no texts are closely followed, but definitions come from [3], [10], and [16].

We open with a brief historical overview of the context in which sheaf cohomology and derived functors arose. The purpose is to motivate what at first seems like a grab bag of esoteric machines. It is based on the historical articles [5], [12], and [17], as well as the original Tohoku paper [8] and letters between Serre and Grothendieck [6].

In §1, we introduce the very basics of *sheaves*, with examples.

In §2, we define an *abelian category*. The definition itself is preceded by a dissection of the categorical properties of Mod_R which make homological algebra "work," so that each stipulated property of an abelian category is justified.

In §3, we define the *derived functors* of a functor between abelian categories. Again, attention is paid to motivate the definition, under the premise that one wants to measure how a functor breaks exactness in a way resembling a cohomology theory. This exposition is partly based on that in [9, Chapter 2, §2], although substantially simplified.

In §4, we define *sheaf cohomology* using derived functors. We first introduce the category of *sheaves of modules* and review its abelian structure. Noting the discrepancy between kernels and cokernels in this category, it starts becoming clear *why* derived functors should be useful for studying local-global problems. In the process, sheaf cohomology is defined as the derived functor of the *global sections functor*.

Historical Notes

While prisoner at an officer camp in Austria throughout World War II, Jean Leray posed as an algebraic topologist so that his expertise in analysis would not volunteer him to aid the Nazi War Machine. The ideas he conceived there would prove important, at times totally indispensable, to the course of algebraic topology and modern complex and algebraic geometry. Namely, he defined *sheaves* and their *sheaf cohomology*.

Around the same time, algebraic topology was seeing an organizational problem. (Co)homology theories abound, absent uniform foundations. Mac Lane and Eilenberg introduced category theory as a language to express ideas in algebraic topology. Then during the 1950-51 Séminare Cartan, more foundational work would be done to expound on and unify the algebraic constructions being used ad hoc to study specific (co)homologies. These ideas collated into H. Cartan and Eilenberg's influential *Homological Algebra* (1956), where the subject got its name.

This homological algebra closely resembled ours today. A glaring limitation, however, is that Cartan and Eilenberg restricted their functors to modules. This was largely superficial. Yet a generalization had not been fully realized, partly since it was not clear when homological algebra could be carried out in other categories.[1]

Enter Grothendieck. Anticipating *Homological Algebra* but without access to it, he developed the subject himself. The result was his landmark Tōhoku paper, which begins with a development of the *abelian categories* in which homological algebra makes sense, based on discussions at Séminare. An upshot was that *derived functors*, originally defined over modules, could be used to formulate a notion of cohomology in any abelian category.

Yet the development of this machinery constituted only about a third of Tōhoku. Indeed, Grothendieck was not fixated on the generalization itself, rather its use in taming a particular abelian category: Leray's sheaves. Sheaves and their cohomology had facilitated several recent mathematical successes—Serre had just demonstrated their unusual effectiveness for studying varieties, and Cartan had done something similar for complex manifolds. In Tōhoku, Grothendieck fit sheaves into the framework of abelian categories, streamlining their use and formulation. All this pushed an installment of sheaf-theoretic and cohomological methods into algebra, number theory, and geometry.

1. Presheaves and Sheaves

Sheaf cohomology detects obstructions to the process of stitching together local constructions into global ones. To get there, we must make precise what we mean by 'local' and 'global' constructions and organize them in a way that is vulnerable to homological algebra. For this we use *sheaves* and *presheaves*.

A (pre)sheaf tracks data (groups, rings, modules, ...) attached to the open sets of a topological space with a notion of locality relative to that space. Relations between data are analogous to the covering relations between their underlying open sets. The extent to which this analogy holds can vary; a presheaf does the bare minimum, demanding just functoriality with respect to inclusions.

Definition 1.1. Let X be a topological space, D a category. A D-*valued presheaf* $\mathcal{F}$ on X is a contravariant functor $\mathcal{F} : \mathsf{Open}(X) \to D$, whose source is the category consisting of open subsets of X and their inclusions. Equivalently, it consists of the following data.

(1) For each open set U of X, an object $\mathcal{F}(U)$. Elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U. Elements of $\mathcal{F}(X)$ are called *global sections*.
(2) For each inclusion of open sets $V \subseteq U$, a morphism $r_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$. This assignment must occur in such a way that $r_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ and if $W \subseteq V \subseteq U$ then $r_{WU} = r_{WV} \circ r_{VU}$. These are called *restriction morphisms*. In context, we use the more familiar notation $|_V$ in place of r_{VU}.

Provided a presheaf and the data it specifies on an open set, we may partially[2] recover the data of subsets via restrictions. In the reverse, another desirable and vaguely geometric property—which a presheaf may or may not have—is that the

[1]David Buchsbaum, then a doctoral student of Mac Lane, refined a definition of Mac Lane's for categories similar to Mod_R and showed how the theory in *Homological Algebra* translated practically verbatim across them [4]. This happened sometime while the text was being written, and its appendix includes Buchsbaum's work. But the full potential of this generalization would only be realized by Grothendieck, who independently redeveloped much of Buchsbaum, Cartan, and Eilenberg's work.

[2]Even for sheaves, restrictions need not be surjective.

data of an open set be recoverable from the data of an open cover. A *sheaf* is a presheaf with this property.

Definition 1.2. A D-*valued sheaf* is a D-valued presheaf satisfying two axioms.

(1) (Identity) Let $\mathcal{U}$ be any open covering of any open set U. If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each $U_i \in \mathcal{U}$, then $s = t$.
(2) (Gluing) Let $\mathcal{U}$ be any open covering of any open set U. If one can choose a section s_i from each U_i such that s_i, s_j agree when restricted to $U_i \cap U_j$ for each i, j, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i.

Remark 1.3. Usually, the target category of (pre)sheaves is Set, Ab, CRing, or Mod_R. In these cases, we get *(pre)sheaves of sets, abelian groups, commutative rings,* or *modules,* respectively.

Definition 1.4. A *morphism of presheaves* is a natural transformation of the respective functors. In other words, a morphism of presheaves φ from $\mathcal{F}$ to $\mathcal{G}$ consists of maps $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each U such that whenever $V \subseteq U$, the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "r_{VU}"'] & \mathcal{G}(U) \arrow[d, "r'_{VU}"] \\
\mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

commutes. A *morphism of sheaves* is defined the same way.

Thus we may speak of the *category of (pre)sheaves* (with values in some chosen category) on a topological space X. We denote the category of (pre)sheaves of abelian groups by $\mathsf{PSh}(X)$ and $\mathsf{Sh}(X)$, respectively.

Example 1.5 (Sheaf of continuous functions)**.** Let X be a topological space. The assignment $U \mapsto \{\text{continuous } f : U \to \mathbb{R}\}$ is a sheaf of rings. Here, one can replace 'continuous' with C^r or C^∞.

Example 1.6 (Presheaf fixed at two points)**.** Let S^1 denote the circle, $p \neq q$ distinct points on it. The assignment $U \mapsto \{\text{ continuous } f : U \to \mathbb{R} \text{ such that } f(p) = f(q)\}$ is a presheaf on S^1. Note that the condition that $f(p) = f(q)$ is vacuous unless $p, q \in U$. Covering S^1 by two open sets U, V, neither containing both p and q, we have sections of U and V agreeing on overlap but disagreeing at p and q; these do not glue into a global section, hence the assignment is not a sheaf.

Example 1.7 (A constant presheaf)**.** Let X be a topological space, G a nontrivial abelian group, and $\mathcal{F}$ the assignment $U \mapsto G$. This $\mathcal{F}$ is a presheaf, the restrictions being identity homomorphisms. To see whether $\mathcal{F}$ is a sheaf, cover the empty set by the empty union, i.e. with no sets. Vacuously, two distinct elements of $\mathcal{F}(\emptyset) = G$ fulfill the identity axiom's hypothesis, but we took them to be distinct; therefore the identity axiom does not hold and $\mathcal{F}$ is not a sheaf.

Example 1.8 (Presheaf of constant functions)**.** Let $\mathcal{F}$ be a presheaf on $X = (-\infty, 0) \cup (0, \infty)$ assigning to each open set U the ring of constant functions $f : U \to \mathbb{R}$. Let $I = (-\infty, 0)$ and $J = (0, \infty)$ have the usual topologies. Any two non-equal constant functions $s \in \mathcal{F}(I)$ and $t \in \mathcal{F}(J)$ vacuously agree on $I \cap J = \emptyset$, yet s and t do not glue into a constant function of X. Thus $\mathcal{F}$ is not a sheaf.

Example 1.9 (Presheaf of bounded functions)**.** Let U be an arbitrary open subset of $\mathbb{R}$ and $\mathcal{F}$ the assignment $U \mapsto \{\text{bounded } f : U \to \mathbb{R}\}$. This is a presheaf on $\mathbb{R}$, with restriction maps being restrictions of domain. Take any unbounded $g : \mathbb{R} \to \mathbb{R}$; restricted to the parts of an open covering of $\mathbb{R}$ by bounded intervals, one gets bounded $\mathbb{R}$-valued functions, hence sections of $\mathcal{F}$. Yet they glue into g, which is not a global section. Thus $\mathcal{F}$ is not a sheaf.

Remark 1.10. We generally want to work with sheaves, but even simple presheaves often fail the sheaf axioms, as the examples show. Fortunately, presheaves and sheaves enjoy a special relationship: there exists a left adjoint to the inclusion $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$. This allows one to canonically construct, for each presheaf, its 'best approximation' as a sheaf.

Strictly speaking, this adjunction is not needed to define sheaf cohomology. But its development is a detour through useful and illuminating theory, and gives an important geometric perspective of sheaves through *espace étalé*.

2. Abelian Categories

The prototypical stomping ground for homological algebra is the category Ab of abelian groups, or generally the category Mod_R of modules over a commutative ring R. One of Grothendieck's motivations for studying abelian categories was to show that Ab and Mod_R, and crucially $\mathsf{Sh}(X)$, are instances of the same thing, over which one can do most homological algebra generally. This streamlines the theory and reveals the essential structure of the situation.

Before stating the definition, let us ruminate on our homological wants and needs. Our models are Ab and Mod_R. Both have a *zero object*, those being the trivial group and zero module, respectively. Then one can define *kernels* and *cokernels*, and they exist and behave as we expect them to (e.g. there is a snake lemma, or a first isomorphism theorem.)

There are also some subtler facts we enjoy in Ab and Mod_R worth explicating. Firstly, morphisms between objects have an abelian group structure; for any $f, g : X \to Y$ one can form their sum $(f + g) : X \to Y$, and this operation has the usual properties. Furthermore, morphisms compose bilinearly, meaning

$$h \circ (f + g) = h \circ f + h \circ g, \text{ and}$$
$$(f + g) \circ h = f \circ h + g \circ h.$$

Getting categorical, these two facts together imply finite coproducts (direct sums) and finite products (direct products), defined by their universal properties, coincide when either exists [11]. When either exists, we may describe $f + g$ using the properties of (co)products: if $f, g : X \to Y$, we have commutative diagrams

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\exists! q"] \arrow[dr, bend left, "g"] & \\
Y & Y \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "i_1"] \arrow[dr, bend right, "\mathrm{id}_Y"'] & Y \coprod Y \arrow[d, dashed, "\exists! p"] & Y \arrow[l, "i_2"'] \arrow[dl, bend left, "\mathrm{id}_Y"] \\
 & Y &
\end{tikzcd}$$

Here, q is induced by f and g, and p by id_Y. Since $Y \times Y \cong Y \coprod Y$, we refer to them unambiguously as $Y \oplus Y$, and we treat $p \circ q$ as a map from X to Y. It turns out that

$$f + g \cong p \circ q.$$

Stipulating the bilinearity of morphisms and the group structure of hom-sets, this construction of $p \circ q$ still requires the existence of the finite (co)product $Y \oplus Y$. All finite (co)products exist in Ab and Mod_R, and this is another property we ask of abelian categories.

Definition 2.1. Consider the following axioms a category may satisfy.

- (PA1) Every hom-set has an abelian group structure.
- (PA2) (PA1), and compositions of morphisms are bilinear with respect to the group structure of hom-sets.
- (A) (PA2), and it has all finitary (co)products. As a consequence of (PA2), we can refer to (co)products synonymously as *biproducts*.
- (AB0) It has a *zero object*. This is an object with exactly one morphism to and from every other object.
- (AB1) (AB0), and it has all *kernels* and *cokernels*. A kernel of a morphism $f : X \to Y$ is an object K and a map $k : K \to X$ such that $f \circ k = 0$, and such that it is universal in that respect. A cokernel is dual.
- (AB2) (AB1), and for every morphism $f : X \to Y$ its coimage (kernel of cokernel) and image (cokernel of kernel) are isomorphic. In particular, the canonical morphism from the coimage to the image is an isomorphism.

A category satisfying (PA2) is called *preadditive*. A category satisfying (A) is called *additive*. A category satisfying (A) and (AB2) is called *abelian*.

These axioms are enough to recast the language of homological algebra over modules to any abelian category. In particular, one has general *exactness*, *complexes*, *chain homotopies*, and *cohomology objects*. Standard facts generalize too, e.g. the snake lemma. We leave the details to standard homological algebra texts, see for instance [8] or [13]. To read the rest of this paper, it suffices understand these ideas in the context of modules or groups.

Remark 2.2. In fact, in many cases it is *formally* sufficient to understand homological algebra in the context of modules. This is the *Freyd-Mitchell embedding theorem*. It states that for each small abelian category, there exists some ring R such that there is a full, faithful, exact functor from that category to Mod_R. Hence, diagram-chasing can be a legitimate proof technique, even if the category of interest has no such concept. For details, see [7].

3. Derived Functors

A fundamental problem in algebraic topology is to understand how certain complexes fail to be exact. Since exactness has meaning in any abelian category, we may ask how *functors between abelian categories* fail to preserve exactness. Amazingly, there is a universal way to quantify this failure, and we call the quantifier a *derived functor*. Derived functors give a unified approach to many important (co)homological theories; for example, the Ext/Tor groups and the sheaf cohomology groups. Our plan is to take the derived functor of the *global sections sheaf* to produce the latter.

3.1. **Motivation.** The initial oddity is that many important functors fail to preserve exact sequences in a small way. Namely, for a functor $\mathcal{F}$ between abelian categories, it may be that

$$0 \to A \to B \to C \to 0 \text{ and}$$

$$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \tag{3.1}$$

are exact in A and B, respectively, but not necessarily so if one completes (3.1) into a short sequence by appending $\to 0$.

Definition 3.2. A functor $\mathcal{F}$ between abelian categories is called *left exact* if for every short exact sequence $0 \to A \to B \to C \to 0$ in the source, the sequence $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$ is exact in the target. One defines a *right exact functor* similarly. An *exact functor* is both left and right exact.

To measure how $\mathcal{F}$ fails to be exact, we look for an extension of (3.1) into a long exact sequence. (If $\mathcal{F}$ is exact, we may simply extend it by $\to 0$'s.) Since there are often many silly ways to extend (3.1), we look specifically for an extension that resembles cohomology. We also hope for a universal such extension.

Desirable Properties. Let $\mathcal{F} : \mathsf{A} \to \mathsf{B}$ be left exact. Consider the following properties a sequence of functors $\{R^i\mathcal{F} : \mathsf{A} \to \mathsf{B}\}_{i\geq 0}$ may have.

(1) There is a natural isomorphism $R^0\mathcal{F} \cong \mathcal{F}$.
(2) For every SES $0 \to A \to B \to C \to 0$, there are morphisms $\delta : R^n\mathcal{F}(C) \to R^{n+1}\mathcal{F}(A)$ so that the following sequence is exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(A) \arrow[r] & \mathcal{F}(B) \arrow[r] & \mathcal{F}(C) \arrow[dll, "\delta"'] \\
& R^1\mathcal{F}(A) \arrow[r] & R^1\mathcal{F}(B) \arrow[r] & R^1\mathcal{F}(C) \arrow[dll, "\delta"'] \\
& R^2\mathcal{F}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

(3) Morphisms between SES in A functorally induce morphisms between the LES from (1) in B.
(4) (Universal Property) If $\{\tilde{R}^i\mathcal{F}\}$ is any other sequence of functors fulfilling (1)-(3), then there is a natural transformation $R^i\mathcal{F} \implies \tilde{R}^i\mathcal{F}$.

Hereafter suppose $R^i\mathcal{F}$ has the above properties. We can glean a constructive definition of $R^i\mathcal{F}$ by looking at the objects A for which $R^i\mathcal{F}(A) = 0$ for all i.

Definition 3.3. Let $\mathcal{F}$ be left exact, $R^i\mathcal{F}$ satisfying properties (1)-(4) as above. An object A is called $R^i\mathcal{F}$*-acyclic* if $R^i\mathcal{F}(A) = 0$ for all i. For an object X, an *acyclic resolution of* X is an exact sequence

$$0 \to X \to A^0 \to A^1 \to \cdots$$

Where the A^i are acyclic.

Lemma 3.4 (Lemma 2.3.6, [9]). *Let $\mathcal{F}, R^i\mathcal{F}$ be as in Definition 3.3. For an object X and any acyclic resolution $A^\bullet$, there is an isomorphism*

$$R^i\mathcal{F}(X) \cong H^i(\mathcal{F}(A^\bullet)).$$

Sketch of Proof. The proof is by induction. Noting that $A^\bullet$ is acyclic, one looks at the long exact sequence of derived functors. Exactness properties apply, and the isomorphism is obtained by routine homological algebra.

Thus, the image under $\mathcal{F}$ of any $R^i\mathcal{F}$-acyclic resolution of X has cohomology isomorphic to $R^i\mathcal{F}(X)$. But acyclic objects are defined relative to $R^i\mathcal{F}$, so this seems circular. A crucial observation about *split sequences* and *injective objects* nevertheless makes it work.

Definition 3.5. We call a short exact sequence $0 \to A \to B \to C \to 0$ *split* if there is an isomorphism $f : B \to A \oplus C$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & B \arrow[r] \arrow[d, dashed, "f"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & A \oplus C \arrow[r, "p"'] & C \arrow[r] & 0
\end{tikzcd}$$

Definition 3.6. An object I in a category is called *injective* if for every morphism $f : X \to I$ and every monomorphism $g : X \to Y$, there exists a morphism $h : Y \to I$ such that $f = h \circ g$.

Lemma 3.7. *If* $0 \to I \to B \to C \to 0$ *is short exact and* I *is injective, then it splits.*

Lemma 3.8. *Left and right exact functors take split exact sequences to split exact sequences.*

Now, given an exact sequence $0 \to I \to B \to C \to 0$ with I injective, it must split, and hence left exact $\mathcal{F}$ will take it to an exact sequence. But $R^i\mathcal{F}$ measures deviation from exactness and is universal in this sense, implying $R^i(I) = 0$ for $i > 0$. Thus injectives are acyclic irrespective of $\mathcal{F}$. For this reason, we ought to compute $R^i\mathcal{F}$ with injective resolutions, but an object need not have such a resolution. A reasonable assumption rules out this possibility.

Definition 3.9. A category is said to *have enough injectives* if every object X admits a monomorphism into an injective object.

If an abelian category has enough injectives, then any object X has an injective resolution obtained by inductively collating the maps induced (à la the defining property of injective objects) by canonical quotient maps into quotient objects.

Example 3.10. Injective modules were introduced by Baer in a 1940 paper [1]. Of primary interest to him was the fact that injective modules are a direct summand of any containing module. In the paper, Baer proved that for a ring R, the category Mod_R has enough injectives.

3.2. **Construction and Properties.** The discussion above motivates the following definition.

Definition 3.11. Let $\mathcal{F} : \mathsf{A} \to \mathsf{B}$ be a left exact functor whose source has enough injectives. For each object $X \in \mathsf{A}$, fix an injective resolution $0 \to X \to I^0 \to \cdots$ and consider the sequence

$$0 \to \mathcal{F}(I^0) \to \mathcal{F}(I^1) \to \cdots \tag{3.12}$$

Obtained by applying $\mathcal{F}$ and composing the maps $0 \to \mathcal{F}(X) \to \mathcal{F}(I^0)$. Define the i-th *right derived functor of* $\mathcal{F}$ at X as the cohomology of (3.12) at the i-th spot, i.e.

$$R^i\mathcal{F}(X) \stackrel{\text{def}}{=} \frac{\ker(\mathcal{F}(I^i) \to \mathcal{F}(I^{i+1}))}{\operatorname{im}(\mathcal{F}(I^{i-1}) \to \mathcal{F}(I^i))}.$$

Theorem 3.13. *Let $\mathcal{F} : \mathsf{A} \to \mathsf{B}$ be as in Definition 3.11.*

(1) Different choices of injective resolutions of objects in A yield naturally isomorphic right derived functors $R^i\mathcal{F}$.
(2) Morphisms $f : X \to Y$ naturally induce morphisms $R^i\mathcal{F}(X) \to R^i\mathcal{F}(Y)$, so that the right derived functors are functors.
(3) There is a natural isomorphism $\mathcal{F}(X) \cong R^0\mathcal{F}(X)$.
(4) For I injective, $R^i\mathcal{F}(I) = 0$ for $i \geq 1$.
(5) For each short exact sequence $0 \to A \to B \to C \to 0$ in A, there are connecting morphisms $\delta : R^i\mathcal{F}(C) \to R^{i+1}\mathcal{F}(A)$ making the following sequence is exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(A) \arrow[r] & \mathcal{F}(B) \arrow[r] & \mathcal{F}(C) \arrow[dll, "\delta"] \\
& R^1\mathcal{F}(A) \arrow[r] & R^1\mathcal{F}(B) \arrow[r] & R^1\mathcal{F}(C) \arrow[dll, "\delta"] \\
& R^2\mathcal{F}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

(6) Furthermore, the connecting morphisms are such that for any morphism between SES in A, the functors $R^i\mathcal{F}$ commute with the connecting morphisms. In effect, morphisms of SES functorally induce morphisms between the LES from (5).

Sketch of Proof.

(1) In general, a morphism between objects in an abelian category lifts to a morphism between resolutions of those objects, i.e. a collection of maps between objects in a resolution commuting with the maps in the resolutions. For two resolutions of the same object, the identity morphism induces a pair of morphisms between resolutions which constitute a homotopy equivalence. Chain homotopic complexes have isomorphic cohomology.
(2) The lifting described in (1) is unique up to homotopy. The resulting chain map gives the induced map on right derived functors.
(3) Since $\mathcal{F}$ is left exact, $0 \to \mathcal{F}(X) \to \mathcal{F}(I^0) \to \mathcal{F}(I^1)$ is exact.
(4) For I injective, the sequence $0 \to I \to I \to 0$ is injective, so that $R^i\mathcal{F}(I) = 0$ for $i > 0$.
(5) See [3, Theorem 2.17] and the discussion afterward.
(6) Same as (5).

4. Sheaf Cohomology as a Derived Functor

We set out to measure how local constructions fail to amass into global ones. This is not unlike asking how the (functorial) process "map a sheaf to its global sections" fails to be exact, with sheaves the ambient category. In fact, this is exactly the right question, and our homological tools from §2-3 will facilitate an answer. But we need a suitable abelian category and *global sections functor*.

4.1. **The Abelian Structure of $\mathcal{O}_X$-Mod.** Let D be a category. Recall from §1 that D-valued sheaves on a topological space form a category. In general, this category is not abelian, but it is so for a particularly important and flexible class of sheaves, namely *sheaves of $\mathcal{O}_X$-modules*. These include sheaves of abelian groups as a special case. They form a category in the most straightforward way.

Definition 4.1. Let X be a topological space and $\mathcal{O}_X$ a sheaf of commutative rings on it. We call the pair $(X, \mathcal{O}_X)$ a *ringed space* and $\mathcal{O}_X$ its *structure sheaf*.

Definition 4.2. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules*, or simply an $\mathcal{O}_X$-module, is a sheaf of abelian groups $\mathcal{F}$ fulfilling the following properties.

(1) For each open $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.
(2) For every inclusion of open sets $V \hookrightarrow U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with module structures: for $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$, the restriction map r satisfies $r(fs) = r(f)r(s)$.

Definition 4.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Denote by $\mathcal{O}_X$-Mod the category of $\mathcal{O}_X$-modules. Its objects are $\mathcal{O}_X$-modules and its morphisms are sheaf morphisms with the additional requirement that their constituent maps $\mathcal{O}_X(U) \to \mathcal{O}_Y(U)$ be module homomorphisms for all open U.

Theorem 4.4. *The category $\mathcal{O}_X$-Mod is abelian. Furthermore, it has enough injectives.*

We rinse our hands of the homological-categorical details implicit in this theorem; details can be found in [3] or [15, 01AF]. But let us review the basic abelian structure of $\mathcal{O}_X$-Mod: the *zero $\mathcal{O}_X$-module* is the one assigning the trivial module to every open set. The *sum of two morphisms of $\mathcal{O}_X$-modules* adds the two maps induced on modules. The *kernel of a morphism* takes the kernel of the induced module homomorphisms.

On the other hand, cokernels are less straightforward. Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules, the assignment $U \mapsto \operatorname{coker}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ does not generally define a sheaf, only a presheaf. The category lover is saved: one *sheafifies*[3] this presheaf into a sheaf that functions as a perfectly good cokernel. But the process has perturbed the definition of a cokernel in $\mathcal{O}_X$-Mod. Consequently, *surjective sheaf morphisms* are more nuanced than *injective sheaf morphisms*.

Definition 4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules.

- One says φ is *injective* if for every open set $U \subseteq X$, $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective as a module homomorphism.
- One says φ is *surjective* if for every open set $U \subseteq X$, every section $s \in \mathcal{G}(U)$, and every point $x \in X$, there is an open neighborhood $V \subseteq X$ of x such that $s_V = \varphi_V(t)$ for some section $t \in \mathcal{F}(V)$.

4.2. **Sheaf Cohomology.** Surjectivity of sheaf morphisms only requires that sections lift *locally*. This is crucial, for given a surjection $\phi : \mathcal{F} \to \mathcal{G}$, it becomes possible that a section of $\mathcal{G}(X)$ may not lie in the image of ϕ_X. This models many geometric local-to-global problems, and derived functors apply readily. Consider the short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \longrightarrow 0 \tag{4.6}$$

[3] *Sheafification* is a functor left adjoint to the inclusion of sheaves into presheaves. It produces the 'best approximation' to a presheaf by a sheaf.

We hit this with the *global sections functor* $\Gamma(X, -)$ sending a sheaf to its global sections. This functor is left-exact, so we can take its derived functors and get a long exact sequence

$$0 \to \Gamma(X, \ker \varphi) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow R^1\Gamma(\ker \varphi) \longrightarrow R^1\Gamma(\mathcal{F}) \to \cdots$$

And finally, we define the *sheaf cohomology functors* $H^i(X, -)$ to be these derived functors $R^i\Gamma(-)$. This cohomology describes, e.g. via the long exact sequence, the relationship between local and global sections of sheaves. For example, if $H^1(X, \ker \varphi) = 0$ above, then $\Gamma(X, -)$ preserved surjectivity in (4.6), so every global section of $\mathcal{G}$ lifts to a global section of $\mathcal{F}$.

Remark 4.7. Computing sheaf cohomology takes some work. The definition by injective resolutions is not used directly. One feasible option is to calculate a space's *Čech cohomology*, which proceeds by combinatorial constructions on open covers. For paracompact spaces, Čech and sheaf cohomology agree [13, Theorem 6.88]. Čech cohomology outdates derived functors, and some sources define sheaf cohomology as Čech cohomology, e.g. Serre's FAC [14]. Čech cohomology is treated extensively in literature, see for instance [2], [3], [13], or [14].

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