Sheaves and Sheaf Cohomology

Reading group notes

July 16, 2022

Abstract

Notes from a reading group at the University of Chicago 2022 REU. The broad goal is to understand some basics regarding sheaves and sheaf cohomology, and to see applications of these and any additional theory (derived functors) we develop. As to the audience's 'level': some of the participants have not seen singular homology of spaces, some have read Hartshorne, ... It will be interesting to see how this goes.

At the very least, everyone will have gotten biscuits and coffee out of this reading group.

Contents

0.1 (Week 2) June 22 - Presheaves and Sheaves

Reading: my paper [1].

First meeting. Opened with some history. (History motivates; see paper.)

Q: What is a sheaf? Furthermore, what (and I cannot stress this enough) *is* a sheaf?

Let X be a topological space. Define $\mathrm{Op}(X)$:= category of its open sets, morphisms given by inclusions.

__WANT:__ A systematic way to 'track' the 'constructions' possible on open sets. E.g., given an open set U, what is the set of cts functions $U \to \mathbb{R}$? How do these sets vary as U varies? How are the resulting sets related, w.r.t. the relation of their underlying open sets? (Try replacing 'cts function' with: C^∞ functions, k-forms, exact k-forms, sections of a vector bundle, solutions to a the PDE arising from a fixed vector field, ...)

Cts functions are the prototypical notion of a 'construction.' Among other things, they have a nice property: if $U \hookrightarrow V$ then a cts $f : V \to \mathbb{R}$ restricts to a cts $f|_U : U \to \mathbb{R}$. (This answers 'how are the resulting sets related?'–they are related by a *restriction map.*)

Any notion of 'construction' should have this restriction property. And this restriction property was just (contravariant) functoriality w.r.t. inclusions.

Definition. A *presheaf* is a contravariant functor $\mathcal{F} : \mathrm{Op}(X) \to \mathsf{D}$, where D is any (concrete) category. Usually it will be sets, rings, abelian groups, ... We call the image of an inclusion under $\mathcal{F}$ a *restriction.*

So a presheaf describes 'constructions' that come with restrictions. Continuous functions have more than just restrictions, though. Two properties in particular stand out (or rather, are so obvious that the need to identify them stands out):

1. if $f_1 : U_1 \to R$ and $f_2 : U_2 \to R$ agree on intersection, then there exists a global $f : U_1 \cup U_2 \to R$ that restricts to f_1 or f_2.
2. If $f, g : U_1 \cup U_2 \to R$ agree when restricted to U_1 or U_2, then $f = g$.

Generalizing (1) and (2) to arbitrary covers leads us to define a sheaf.

Definition. A *sheaf* is a presheaf satisfying two axioms.

1. (Identity) Let $\{U_i\}$ be any open covering of any open set U. If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each U_i, then $s = t$.
2. (Gluing) Let $\{U_i\}$ be any open covering of any open set U. If one can choose a section s_i from each U_i such that s_i, s_j agree when restricted to $U_i \cap U_j$ for each i, j, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i.

__EXAMPLES__–important, see paper, end of §1.

Remark. (Pre)sheaves form categories $\mathsf{PSh}(X)$, $\mathsf{Sh}(X)$. Objects are (pre)sheaves, morphisms are natural transformations. The inclusion functor $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ has a left adjoint called *sheafification.* Thus, sheafifying a presheaf gives the 'best' sheaf approximate. The exact construction of sheafification is not widely useful; people (algebraic geometers) usually just need to know the universal property relating a presheaf to its sheafification. (The one described by adjunctions.)

0.2 (Week 2) June 26 - Abelian Categories I

Still reading [1].

Opened with historical refresher: homological algebra had just systematized our understanding of modules, at least in the context of their use in algebraic topology. This clearly worked for objects besides modules, but a rigorous abstraction was not made.

In particular, *sheaves* had 'homological algebra-like' properties, e.g. kernels, quotients. And sheaves had JUST effected advances in complex/algebraic geometry (Serre, Cartan, ...) The big question: can we figure out how to do homological algebra generally, fully fit sheaves into their framework, then recruit the techniques of algebraic topology (cohomology!) to study sheaves? Short answer: yes.

Today, we want to identify what makes homological algebra work. Or rather, where it works.

The base case is Mod_R, the category of R-modules–this is where we first understood homological algebra. Let's specialize to Ab, the category of abelian groups. What are its properties?

1. Given two groups G and H, the set $\mathrm{Hom}(G, H)$ is an abelian group.
2. With respect to the group structure above, morphisms compose bilinearly; $h \circ (f + g) = h \circ f + h \circ g$, and similarly with $(f + g) \circ h$.
3. It turns out that (1) and (2) imply that the (finite) product and coproduct of two groups are isomorphic, *if either exists*. (Here, I mean the categorical (co)product; in strictly group-theoretic terms, these are the direct sum/product.) Assuming they exist, we have a natural categorical description of the addition law on hom-sets (that's good!), see the reading p. 5-6, starting at "Getting categorical, ..." And it turns out that Ab *does* have all finite products and coproducts. (That's the property.)
4. It has a *zero object*, to mean an object with a unique morphism to AND from all other objects. (It's the trivial group.) This gives rise to e.g. *zero morphisms*.

Then the Reg closed and some of us got dinner. We'll finish listing properties next time.

0.3 (Week 3) June 29 - Abelian Categories II

Reading: my paper [1]. (Supplementary: Here's a reference for the basics of singular homology. Here's a basic example of 'use singular homology to study a space.' Sheaf cohomology makes an appearance in the second link.)

Picking up where we left off—what are the good properties of Ab, insofar as homological algebra is concerned?

5. Kernels and cokernels of morphisms are (sub)groups.
6. The first isomorphism theorem: if $f : G \to H$ is a hom, then $G/\ker f \cong \operatorname{im} f$. Another name for $G/\ker f$ is the *coimage* of f. This notation is good to know.

Facts (1)-(6) let us do 'homological algebra' with abelian groups–we can talk about chain complexes, quotients, kernels, etc. Now we reformulate these properties for arbitrary categories. This requires we 'categorify' certain terms above. Here's what we get:

Definition. Consider the following properties a category may have.

(PA1) Every hom-set has an abelian group structure.

(PA2) (PA1), and composition is bilinear with respect to hom-set addition law.

(A) (AB2), and it has all finitary products and coproducts. **Consequences**: we mentioned that this implies the (co)product of two objects coincide. We also mentioned that this inicidence gives us a categorical construction of the addition law on hom-sets.

(AB0) It has a *zero object.*

(AB1) (AB0), and it has all *kernels* and *cokernels.* **Definitions**: the *kernel* of a morphism $f : X \to Y$ is the equalizer of f and the zero map $X \to Y$. Equivalently, it is a map $k : K \to X$ such that (i) $f \circ k$ is the zero map, and (ii) for any map $k' : K' \to X$ such that $f \circ k'$ is zero, there is a unique map $K' \to K$ making that whole diagram commute. Dually, the *cokernel* of f is the coequalizer of f and the zero map $X \to Y$.

(AB2) *Images* and *coimages* are isomorphic. **Definitions**: Given f, the *image* of f is defined is the kernel of its cokernel. Dually, the *coimage* of f is the cokernel of its kernel. Actually, I think the isomorphism is 'canonical': by the definitions of (co)images, there appears a map $\operatorname{coker} \ker f \to \ker \operatorname{coker} f$, which gives the isomorphism(?)

Definitions: A category satisfying (PA2) is called *preadditive.* A category satisfying (A) is called *additive.* A category satisfying (A) and (AB2) is called *abelian.*

Abelian categories have homological algebra!

0.3.1 Digression: homological algebra?

At this point, I should stop saying 'homological algebra' without any indication of what it is. Especially since some of us might not know examples of (co)homology. I'll introduce *singular (co)homology.* It is a good 'toy example' witnessing the emergence of algebra in the study of spaces.

Let X be a topological space. Define $C_n(X) := FAb\{\text{continuous } \sigma : \Delta^n \to X\}$.[1] The n-simplex Δ^n has faces, so $\sigma \in C_n(X)$ can be restricted to a face, which defines a map $\Delta^{n-1} \to X$. Define a homomorphism $\partial : C_n(X) \to C_{n-1}(X)$ by

$$\sigma \mapsto \sum (-1)^i \sigma_{\text{its } i\text{-th face}}.$$

That $(-1)^i$ accounts for orientation, which makes everything work. We can forget about it.

Now we look at the following 'chain':

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \to \cdots \to C_1 \xrightarrow{\partial} C_0.$$

Note that $\partial^2 = 0$; in this case, we call the chain a *chain complex of abelian groups.* Since $\partial^2 = 0$, we know $\operatorname{im}\partial \subseteq \ker\partial$. So we may define $H_n(X) := \ker \partial / \operatorname{im} \partial$. These are the *homology groups* of X.

We wish to know things about $H_n(X)$. It's definition is in terms of that chain complex—*it turns out we can just study abstract chain complexes and learn a lot about $H_n(X)$ as a result.* That's homological algebra.

[1] FAb means 'free abelian group on this set.' And Δ^n is the n-simplex; Google its definition.

*__Example__: one learns that given a pair of spaces $A \subset X$, there is a corresponding 'long exact sequence' (LES) of singular homology groups of A, X, and (A, X). The proof is pure (homological) algebra, making no reference to how C_n is defined. It's an example of a more general fact: given a SES of chain complexes, there is an induced LES in homology. In this case, $A \to X \to X/A$ is a SES of spaces, which we hit with $C_*(-)$ to get a SES of chain complexes, to get the LES in homology. The proof depends on the *snake lemma* and *zig-zag lemma* from hml. algebra.

0.3.2 The furniture of abelian categories

Finally, we have defined abelian categories. They come with:

- A first isomorphism theorem (by assumption).
- Quotients (and 'subobjects').
- (Co)chain omplexes. And since we have quotients, we can define the *(co)homology* of a chain (co)complex.
- Since we have (co)kernels, we can talk about *exactness*.
- A snake lemma and zig-zag lemma.

This exactly extends the language we used to define and study singular homology to work in other categories (e.g., modules). This corroborates our history lesson: they were coming up with algebra to help them do 'combinatorial topology,' then wanted to seperate and systematize all that algebra.

0.3.3 Upshot

Here's vaguely how we use singular homology:

1. Take a space, (e.g. X)
2. Construct algebraic things from that space that 'knows' something about the topology of X, (e.g. $C_n(X)$, ∂)
3. Use homological algebra to study those 'things,' (e.g. $H_n(X)$, derivation of LES induced by SES)
4. Leverage the relationship between X's topology and the $H_n(X)$ to learn about X. (E.g., show X is not simply connected by showing $H_1(X)$ is nontrivial; show that a 3-manifold has a non-contractible embedded curve if H_2 is nontrivial; this.)

With abelian categories defined, we can be a bit more creative. We may replicate (3) for more interesting 'algebraic things.' As a pointed example, we may try to:

1. Take a space X,
2. Construct a gadget that 'knows' when 'local constructions we care about' are possible and how they patch together. Note: this gadget 'knows' something about X's topology AND about the 'constructions we care about.'
3. See if those gadgets form an abelian category. If they do, study them from the POV of homl. algebra, perhaps deriving algebraic data K from our gadgets.
4. Leverage the relationship between X and the K to learn about X, AND to learn about the 'constructions.'

Of course, by ‘gadgets’ I mean *sheaves!* I hope the two set of steps I gave appear analogous; I tried to get across some of the intuition I developed while trying to learn this stuff the first time around.

Note: steps (4) and (4) are very similar. However, although X's topology tells us about $H_n(X)$, we don't care about $H_n(X)$ past what it says about X; what $H_n(X)$ concretely describes (maps of n-simplices) is not intrinsically significant. But such is not generally the case...

0.4 (Week 4) July 6 - Oops, I Pontificated

Supposed to have covered derived functors today. Instead, I talked too much about singular homology, mistakenly thinking a lot of us did not know about singular homology.

PREVIOUSLY: Defined abelian categories. They had homological algebra. To motivate homl. algebra, we defined and discussed the singular homology $H_n(X)$ of a space X. Pure homological algebra produced useful facts about $H_n(X)$.

0.4.1 A bit more thinking about chains

We should work out one example of 'using homological algebra to do (co)homology.' Let's cover a particularly important property of $H_n(X)$. This is the 'long exact sequence (LES)' I talked about in the last talk.

Proposition 0.1. *Let X be a space, A a subspace. Then the following sequence is exact:*

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots \to H_0(A) \to H_0(X) \to H_0(X,A) \to 0.$$

The maps $i_i : H_i(A) \to H_i(X)$ are induced by inclusion. The maps $j_i : H_i(X) \to H_i(X,A)$ are induced by 'relative inclusion.' The maps $\partial : H_i(X,A) \to H_{i-1}(A)$ are the connecting homomorphisms.

The maps i_*, j_* are God-given and trivial. From definitions it follows that $H_i(A) \to H_i(X) \to H_i(X,A)$ is exact. These constitute 'pieces' of our LES.

But where did ∂ come from, the maps that connect all 'pieces' together? Answer: ∂ arises from pure algebra. Here's how:

Definition. A *chain complex of abelian groups* is a sequence of abelian groups

$$\ldots \xrightarrow[\partial_1]{} G_3 \xrightarrow[\partial_0]{} G_2 \xrightarrow[\partial_{-1}]{} G_1 \to 0$$

Such that $\partial_i \partial_{i-1} = 0$ for all i.

Definition. Let G_* be a chain complex, written as above. Its k-th *homology group* is defined to be $H_k(G_*) := \ker \partial_{k+1} / \operatorname{im} \partial_k$.

Fact. Suppose $0 \to A_* \to B_* \to C_* \to 0$ is an exact sequence of chain complexes. Then there are connecting morphisms of abelian groups $\partial : H_i(C_*) \to H_{i-1}(A_*)$ making the following sequence exact:

$$\ldots \xrightarrow[\partial]{} H_n(A_*) \to H_n(B_*) \to H_n(C_*) \xrightarrow[\partial]{} H_{n-1}(A_*) \to \ldots$$

In the context of singular homology: a pair of spaces $A \subseteq X$ induces an exact sequence of chain complexes $0 \to C_*(A) \to C_*(X) \to C_*(X/A) \to 0$ where $C_*(Y)$ is the chain of groups of n-simplices mapping into Y. Then the fact in this case produces 0.1.

I wrote this out for two reasons:

1. To get us used to some of the language (exactness, kernels, ...)
2. The LES 0.1 is very important in algebraic topology. Given a pair of spaces $A \subseteq X$, many other (co)homology theories have an identical LES to 0.1, with H_i

replaced. *In fact, this is so important that the axioms for a (co)homology theory demand that any pair $A \subseteq X$ give rise to a LES; see Axiom 5.* Soon, we are going to 'look' for a long exact sequence; I discussed them here so as to convince you that this is a natural thing to look for.

0.4.2 Derived functors

A fundamental problem in algebraic topology is to describe when and how complexes fail to be exact. We saw this concretely: the singular homology of a space $H_n(X)$ describes the extent to which n-simplices mapping into X did not arise as the boundaries of $(n+1)$-simplices. Many other (co)homologies are also just concrete exactness measurements, too: simplicial, cellular, de Rahm, ...

Philosophically, (co)homology describes 'obstruction' to exactness:

- If $H_1(X)$ is nontrivial (i.e. the singular chain is not exact at $C_1(X)$), then nontrivial topology of X is obstructing some 1-chain $\sigma \in C_1(X)$ from occurring in the image of $\partial : C_2(X) \to C_1(X)$. In particular, a '1-dimensional hole.'
- If X has trivial topology, then $H_{i>0}(X) = 0$.

(To do: finish recapping today's meeting.)

0.5 (Week 4) July 10 - Proving some things

Well, we *did* do things today. But some people slept in, so we spent time not cutting through content but working through a proof. We'll take it from the top next time, when everyone is present, and I'll present part of the proof we looked at.

0.6 (Week 5) July 14 - Derived Functors

Fix a left-exact functor $\mathcal{F} : \mathsf{D} \to \mathsf{E}$ between abelian categories. Recall what this means: for any SES $0 \to A \to B \to C \to 0$ in D, the resulting $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$ is exact in D.

0.6.1 Functors of Good Repute

Q: How to quantify the extent to which $\mathcal{F}$ is not fully exact? A: Look for 'canonical' exact extension

$$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to ???$$

(If $\mathcal{F}$ *is* exact, we can just extend by $\to 0$'s.) Q: What should we mean by 'canonical'? A: Take the lead from algebraic topology (also group cohomology, and other places). Here's what we get:

Desirable Properties: We dream of functors $\{R^i\mathcal{F} : \mathsf{D} \to \mathsf{E}\}_{i \geq 0}$ such that:

1. There is a natural isomorphism $R^0\mathcal{F} \cong \mathcal{F}$.
2. For each SES $0 \to A \to B \to C \to 0$ in D, there are morphisms $\delta : R^n\mathcal{F}(C) \to R^{n+1}\mathcal{F}(A)$ such that the following sequence is exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(A) \arrow[r] & \mathcal{F}(B) \arrow[r] & \mathcal{F}(C) \arrow[dll, "\delta"] \\
& R^1\mathcal{F}(A) \arrow[r] & R^1\mathcal{F}(B) \arrow[r] & R^1\mathcal{F}(C) \arrow[dll, "\delta"] \\
& R^2\mathcal{F}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

3. Morphisms of SES in D *functorally* induce morphisms between the LES arising from (2).
4. (Universality AKA Minimality) If $\{R^i\mathcal{F}'\}$ is any sequence of functors satisfying (1)-(3), then there is a natural transformation $R^i\mathcal{F} \to R^i\mathcal{F}'$ for all i.

If (3) is not clear: it says that given two SES

$$0 \to A \to B \to C \to 0, \qquad 0 \to A' \to B' \to C' \to 0$$

And a morphism between them, the colored morphisms on the next page exist so that the whole diagram commutes. (The commutativity is the 'functorial' requirement.)

$$
\begin{tikzcd}
0 \arrow[r] & R^0\mathcal{F}(A) \arrow[r] \arrow[drrrr, bend left, magenta] & R^0\mathcal{F}(B) \arrow[r] \arrow[drrrr, bend left, magenta] & R^0\mathcal{F}(C) \arrow[ddll] \arrow[drrrr, bend left, magenta] & & & & \\
& & & & 0 \arrow[r] & R^0\mathcal{F}(A') \arrow[r] & R^0\mathcal{F}(B') \arrow[r] & R^0\mathcal{F}(C') \arrow[ddll] \\
& R^1\mathcal{F}(A) \arrow[r] \arrow[drrrr, bend left, magenta] & R^1\mathcal{F}(B) \arrow[r] \arrow[drrrr, bend left, magenta] & R^1\mathcal{F}(C) \arrow[ddll] \arrow[drrrr, bend left, magenta] & & & & \\
& & & & & R^1\mathcal{F}(A') \arrow[r] & R^1\mathcal{F}(B') \arrow[r] & R^1\mathcal{F}(C') \arrow[ddll] \\
& \dots & & & & & & \\
& & & & & \dots & &
\end{tikzcd}
$$

0.6.2 Acyclics

Suppose a sequence of functors $\{R^i\mathcal{F}\}$ satisfying (1)-(4) exists. We're going to figure out how to construct it based on its properties.

Definition. An object A in the domain is called $R^i\mathcal{F}$*-acyclic* if $R^i\mathcal{F}(A) = 0$ for $i > 0$. Furthermore, an *acyclic resolution* of X is an exact sequence

$$0 \to X \to A^0 \to A^1 \to \dots$$

Such that all the A^i are $R^i\mathcal{F}$-acyclic.

Proposition 0.2 (Acyclics compute $R^i\mathcal{F}$)**.** *Let X be in the domain, and suppose*

$$0 \to X \to A^0 \to A^1 \to A^2 \to \dots$$

Is an acyclic resolution of X. Then

$$R^i\mathcal{F}(X) \cong H^i(\mathcal{F}(A^\bullet)).$$

Proof. ($i = 0$): Since $\mathcal{F}$ is left-exact, the first few terms of $\mathcal{F}(A^\bullet)$ are exact. Specifically,

$$0 \to \mathcal{F}X \to \mathcal{F}A^0 \to \mathcal{F}A^1$$

Is exact. By exactness, $\mathcal{F}X \cong \ker(\mathcal{F}A^0 \to \mathcal{F}A^1)$. By definition, the RHS is $\cong H^0(A^\bullet)$. By property (1), the LHS is $\cong R^0\mathcal{F}(X)$.

($i = 1$): Now consider the SES $0 \to X \to A^0 \to A^0/X \to 0$, obtained from the acyclic resolution. By (2), this induces a LES:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}X \arrow[r] & \mathcal{F}A^0 \arrow[r] & \mathcal{F}(A^0/X) \arrow[dll] \\
& R^1\mathcal{F}X \arrow[r] & 0 \arrow[r] & R^1\mathcal{F}(A^0/X) \arrow[dll, "\sim"] \\
& R^2\mathcal{F}X \arrow[r] & \dots &
\end{tikzcd}$$

Those zeroes appear since A^0 is acyclic. By exactness, $\mathcal{F}(A^0/X) \to R^1\mathcal{F}X$ is surjective, so by the FIT $R^1\mathcal{F} \cong \mathcal{F}X(A^0/X)/\ker$ which by exactness is $\cong \mathcal{F}(A^0/X)/\mathrm{im}(\mathcal{F}A^0 \to \mathcal{F}(A^0/X))$.

Now, two parts:

(1) notice that A^0/X is the kernel of $A^1 \to A^2$, by the FIT. Therefore, $\mathcal{F}(A^0/X)$ is the kernel of $\mathcal{F}A^1 \to \mathcal{F}A^2$.[2]

(2) notice that $\mathrm{im}(\mathcal{F}A^0 \to \mathcal{F}(A^0/X)) \cong \mathcal{F}A^0/\ker$ by FIT, which by exactness gives us $\cong \mathcal{F}A^0/\mathcal{F}X$. That is precisely the image of $\mathcal{F}A^0 \to \mathcal{F}A^1$.

Therefore, $R^1\mathcal{F}X \cong \ker/\mathrm{im}$ which is equal by definition to $H^1(\mathcal{F}(A^\bullet))$.

($i > 1$) Induction. Suppose $R^k\mathcal{F}(V) \cong H^k(\mathcal{F}(Y^\bullet))$ for all $k < i$ and any acyclic resolution $Y^\bullet$ of V. See that the LES gives us $R^i\mathcal{F}X \cong R^{i-1}\mathcal{F}(A^0/X)$. By assumption,

[2]This is one way to define left-exact functors: they preserve kernels.

$R^{i-1}\mathcal{F}(A^0/X) \cong H^{i-1}(\mathcal{F}Y^\bullet)$, where $Y^\bullet$ is a resolution of A^0/X. And we know such a resolution:

$$0 \to A^0/X \to A^1 \to A^2 \to A^3 \to \dots$$

So let $Y^\bullet$ be the above resolution. The $(i-1)$-th cohomology of $Y^\bullet$ is the i-th cohomology of $A^\bullet$; that is, $H^{i-1}(\mathcal{F}Y^\bullet) \cong H^i(\mathcal{F}A^\bullet)$. So $R^i\mathcal{F}X \cong H^i(\mathcal{F}A^\bullet)$. □

0.6.3 Injectives

We supposed $R^i\mathcal{F}$ existed and Prop. 0.2 gave us a construction of $R^i\mathcal{F}$. However, it requires acyclics, which are defined relative to $R^i\mathcal{F}$, so this is circular. To get a better construction, we will think harder about $R^i\mathcal{F}$-acyclic objects. We will realize there are certain objects which are *always* $R^i\mathcal{G}$-acyclic, for any $\mathcal{G}$.

What does an 'always acyclic' object look like? Recall that $R^i\mathcal{F}$ is meant to measure how $\mathcal{F}$ fails to preserve SES's. In particular, it should be zero on objects which form a SES that is taken to a SES. Let me say that firmly:

Proposition 0.3. *Suppose $J \in \operatorname{Ob} D$ is such that whenever $0 \to J \to M \to N \to 0$ is exact, its image $0 \to \mathcal{F}J \to \mathcal{F}M \to \mathcal{F}N \to 0$ is exact. Then J is $R^i\mathcal{F}$-acyclic.*

Therefore, *if there exists J such that (Property) "$0 \to J \to M \to N \to 0$ is exact $\implies$ the image of that SES under any left exact functor is exact," then J is $R^i\mathcal{G}$-acyclic for any left-exact $\mathcal{G}$.*

So the J with that Property are acyclic, for the derived functors of *any* left-exact functor. On the other hand, we know of a class of SES whose image is always exact:

Definition. A short exact sequence $0 \to A \to B \to C \to 0$ is called *split exact* if there is an isomorphism $f : B \to A \oplus C$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal, "="'] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, equal, "="] & 0 \\
0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Proposition 0.4. *The image of a split-exact sequence under a left or right exact functor is a split-exact sequence.*

Therefore, if we can find an object I such that whenever $0 \to I \to B \to C \to 0$ is exact it is split-exact, then I will be $R^i\mathcal{F}$-acyclic, for any left-exact $\mathcal{F}$, by Prop 0.3.

At this point, maybe you think this is contrived or unnatural. However, these properties were of interest to people near the time when sheaf cohomology and abelian categories were emerging. In 1940, Baer wanted to know about when modules split (as a sum) of other modules. In particular, he sought modules I such that if $I \hookrightarrow M$ then $M \cong I \oplus$ something. Speaking with hindsight, he was essentially asking for a module I such that whenever

$$0 \to I \to M \to N \to 0$$

Is exact, it is split-exact. Baer found his modules, therefore he solved our problem for us.

Definition. An object I is called *injective* if for every morphism $f : X \to I$ and every monomorphism $g : X \to Y$, there exists an $h : Y \to I$ such that $f = h \circ g$.

Proposition 0.5. *If* $0 \to I \to B \to C \to 0$ *is exact and* I *is injective, then it splits.*

In particular, ANY left-exact functor will take $0 \to I \to B \to C \to 0$ to a split-exact sequence. By Prop 0.3, I must be $R^i\mathcal{G}$-acyclic for ANY left exact $\mathcal{G}$.

<u>So:</u> Injectives are always acyclic.

Thus, given left-exact $\mathcal{G}$, the general procedure for computing $R^i\mathcal{G}(X)$ is:

1. Resolve X by injectives.
2. Hit that resolution with $\mathcal{G}$.
3. Take its cohomology.

Using injectives means we don't have to keep track of which objects were $R^i\mathcal{G}$-acyclic vs. $R^i\mathcal{H}$-acyclic for some other $\mathcal{H}$. That's the point.

Now, we need to ensure injective resolutions actually exist. One assumption gives us this.

Definition. An abelian category is said to *have enough injectives* if every object X admits a monomorphism into an injective object.

0.7 (Week 5) July 15 - Michael Barz: ODE's Local Systems, and Monodromy Representations

References: Deligne gave a Harvard course on this, Schnell's course on D-modules, Katz's book *Rigid Local Systems*.

ODE gives representation of fundamental group. Does every representation arise from an ODE?

Riemann asked this. Deligne introduced *local systems* to answer(?) it.

...

An ODE is always locally solvable.

Given solution in small region, does it arise as restriction of a global solution?

Given points p, q where f defined near p, CHOOSE path from p to q and locally solve along path.

Monodromy Thm: Given two paths from p to q, IF they are homotopic, THEN the extensions of any f defined near p along each path will agree at q.

Now consider non-simply connected domains. Specifically, $\mathbb{C}^\times$. Try solving

$$\frac{df}{dz} = \frac{1}{2z} f$$

Can solve this near 1 and -1. And we have two non-homotopic paths from 1 to -1. One goes over 0, one goes under. Extending to solution along these two paths yields different results: extension along top gives negative at -1, extension along top gives positive at -1.

Define REP of $\pi_1(X)$ as follows:

1. Start with ODE $f' = f/2z$.
2. Take basepoint = 1, start with local solution.
3. Take loop γ based at 1.
4. Continue around loop.
5. Get NEW solution near 1 after you go around.
6. This defines a transformation of 'space of local solutions to ODE at 1' to itself, like an action by action of some $M \in GL_1\mathbb{C}$.
7. Think of M as representing loop.

How to formalize using sheaves?

Let $L(f)$ denote our 'ODE operator.'

Definition. *Sheaf of solutions*: Define $\mathcal{F}$ so that $\mathcal{F}(U) := \{f : U \to \mathbb{C} : L(f) = 0\}$.

This is a sheaf because ODE's only need to be solved locally.

Proposition 0.6. *Special property: this sheaf is a local system. A sheaf $\mathcal{G}$ is called a local system if $\exists\{U_i\}$ covering X s.t. G_{U_i} = a constant sheaf indp. of i, restricted to U_i.*

Local existence and uniqueness of solutions to ODES says that *the sheaf of solutions of an ODE is always a local system.*

OK, time to formalize stuff before.

0.7.1 Some sheaf terminology.

Definition. Let $\mathcal{G}$ be a sheaf on X. Define for $p \in X$ the *stalk* of $\mathcal{G}$ at p as

$$\mathcal{G}_p := colim_{p \in U} \mathcal{G}(U).$$

This is equal to

$$\{(s, U) : s \in \mathcal{G}(U), p \in U\}/\sim$$

Where $(s, U) \sim (t, V)$ if there is R containing p such that $s|_R = t|_R$.

Example: If $X = \mathbb{C}$ and $\mathcal{G}$, then $\mathcal{G}_0$ = ring of convergent power series at 0.

...

GOAL: Given a local system $\mathcal{G}$ on X and $x \in X$, define $e : \pi_1(X, x) \to GL(\mathcal{G}_x)$.

Let γ be a loop $[0, 1] \to X$, based at x. We have $\mathcal{G}$ on X. Can we get a related sheaf on $[0, 1]$?

Definition. *Inverse image sheaf.* Define $\gamma^*\mathcal{G}(U)$ to the SHEAFIFICATION of (COLIM of $\mathcal{G}(V)$ over V containing $\gamma(U)$). Michael gave a good 'moral' picture of this. The reason we need to use these ugly things is because maps are not always open.

Proposition 0.7. *Inverse image has nice property: $\gamma^*\mathcal{G}_t = \mathcal{G}_{\gamma(t)}$. I.e., inverse image sheaf commutes with taking stalks. Proof is formal. Needs that sheafification commutes with stalks.*

We have canonical isomorphisms $\mathcal{G}_x \to \gamma^*\mathcal{G}_0$ and $\mathcal{G}_x \to \gamma^*\mathcal{G}_1$. We are interested in the composition:

$$\mathcal{G}_x \to \gamma^*\mathcal{G}_1 \to \gamma^*\mathcal{G}_0 \to \mathcal{G}_x.$$

This composition will be our representation. BUT we need to CONSTRUCT that middle map.

WANT: Let S be a sheaf on $[0, 1]$ where S is a local system. WANT a map $\gamma : S_0 \to S_1$. What is S_0? It consists of germs at 0. For some ϵ we know $S|_{[0,\epsilon)} \cong$some constant sheaf G on $[0, \epsilon)$. So a germ $s \in S_0$ extends to a section $s' \in S([0, \epsilon))$. Extend further to $S([0, 2\epsilon))$ and further ... Compactness lets us do this. We end up with a section $S([0, 1])$. Restrict back to S_1. Finally get a morphism $S_0 \to S_1$.

(Another way to think about this is to realize $S_t \cong S([0, 1])$, which is a special property of local systems on I, and use this to define $S_0 \to S_1$.)

SO: We have defined a function

$$e : \{\text{loops at } x\} \to GL(\mathcal{G}_x).$$

Still want to show e is well-defined up to homotopy. We very briefly sketched the proof. The key fact: *if S is a local system on the square, then $S_t \cong S$(the square).*

(Still about the proof) remember that we have two maps:

$$M_{\gamma_0} : \mathcal{G}_x \to \gamma_0^*\mathcal{G}_0 \to \gamma_0^*\mathcal{G}_1 \to \mathcal{G}_x$$

$$M_{\gamma_1} : \mathcal{G}_x \to \gamma_1^*\mathcal{G}_0 \to \gamma_1^*\mathcal{G}_1 \to \mathcal{G}_x$$

Want a morphism of these two(?) We happen to have an isomorphism $\gamma_0^*\mathcal{G}_0 \cong H^*\mathcal{G}(0,0)$. Here, H is related to a homotopy between two maps γ_1, γ_0. Similarly, $\gamma_1^*\mathcal{G}_0 \cong H^*\mathcal{G}_{(0,1)}$. Make replacements. Show the whole damn thing commutes.

End proof attempt.

0.7.2 Bigger Picture

So far we have done this:

$$ODE's \to LocalSystems \to MonodromyReps$$

Turns out the second arrow has an inverse, giving an equivalence of categories. How? Covering space theory. The idea is: Given a monodromy representation $\gamma : \pi_1(X, x) \to GL_n(\mathbb{C})$, that's like an automorphism of points in universal cover of X, so let L be a sheaf on X defined by

$$L(U) := \{\text{ctss } s : \pi^{-1}(U) \to \mathbb{C}^n \; : \; \gamma \cdot s = \gamma(x)s\} \quad \text{(?I think I made a typo)}$$

Cool fact: We can recover Riemann surfaces from this. Idea: We can't define square root on $\mathbb{C}^\times$. But can always solve ODE's over universal cover. Nw look at monodromy rep e assocatied to $\sqrt{z}$ ODE. Its kernel gives some Riemann surface?

Final Q: What is the category of ODE's? And what is the (obstruction to) an inverse to the arrow ODE's → Local Systems? Something-something-something vector bundles with connection, look at flat connections. What we get is an equivalence of categories:

$$ODE's \leftrightarrow LocalSystems$$

0.7.3 Curtain Call

Then, Michael gave us exercises. Oh, Michael.

Then Judson asked a good question about making everything more complicated, and Michael says that we can make everything more complicated. (D-modules, perverse sheaves, Riemann hypothesis for finite fields, ...)

Then, Michael said analysts actually care about this stuff. For example: the monodromy representation can tell us about the behavior of ODE's near singularities, i.e. are they singular?

Also, very deep implications toward number theory. (Grothendieck-Katz p-curvature conjecture.)

References

[1] Matthew A. Niemiro. *The derived functor approach to sheaf cohomology*. University of Chicago Mathematics REU. 2020.